Separation axioms on enlargements of generalized topologies

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Abstract. The aim of this paper is to characterize the $\kappa_\mu$-closure of any subset $A$ of $X$ and study under what conditions a subset $A$ of $X$ is $g.\kappa_\mu$-closed. We also introduce the notions of $\kappa$-$T_i$ ($i = 0, 1/2, 1, 2$) and study some properties of them.

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1. Introduction

In 2002, Császár [1] introduced the notions of generalized topology and generalized continuity. In 2008, Császár [3] defined an enlargement and construct the generalized topology induced by an enlargement; introduced the concept of $(\kappa, \lambda)$-continuity and $(\kappa_\mu, \lambda_\mu)$-continuity on enlargements. In 2008, Császár [4] defined and studied the notions of product of generalized topologies. In 2010, S. Maragathavalli et al. in [5] studied the $g.\kappa_\mu$-closed sets in generalized topological spaces and gave some characterization and properties. Also V. Renukadevi in [6] gave a characterization of $g.\kappa_\mu$-closed using enlargements. In this paper we characterize the $\kappa_\mu$-closure of any subset $A$ of $X$, compare the sets $c_\kappa$ defined in [3] and $c_{\kappa_\mu}$, study under what conditions a subset $A$ of $X$ is $g.\kappa_\mu$-closed) and introduce the notions of $\kappa$-$T_i$ ($i = 0, 1/2, 1, 2$) and study some properties of them, finally we study some notions related with the product of generalized topologies.
2. Preliminaries

Let $X$ be a nonempty set and $\mu$ be a collection of subsets of $X$. Then $\mu$ is called a generalized topology on $X$ if and only if $\emptyset \in \mu$ and $G_i \in \mu$ for $i \in I \neq \emptyset$ implies $\bigcup_{i \in I} G_i \in \mu$. We call the pair $(X, \mu)$ a generalized topological space on $X$. The elements of $\mu$ are called $\mu$-open sets [1] and the complements are called $\mu$-closed sets. The generalized-closure of a subset $A$ of $X$, denoted by $c_\mu(A)$, is the intersection of all $\mu$-closed sets containing $A$; and the generalized-interior of $A$, denoted by $i_\mu(A)$, is the union of $\mu$-open sets included in $A$. Let $\mu$ be a generalized topology on $X$. A mapping $\kappa : \mu \to P(X)$ is called an enlargement [3] on $X$ if $M \subseteq \kappa M$ ($= \kappa(M)$) whenever $M \in \mu$. Let $\mu$ be a generalized topology on $X$ and $\kappa : \mu \to P(X)$ an enlargement of $\mu$. Let us say that a subset $A \subseteq X$ is $\kappa_\mu$-open [3] if and only if $x \in A$ implies the existence of a $\mu$-open set $M$ such that $x \in M$ and $\kappa M \subseteq A$. The collection of all $\kappa_\mu$-open sets is a generalized topology on $X$ [3]. A subset $A \subseteq X$ is said to be $\kappa_\mu$-closed if and only if $X \setminus A$ is $\kappa_\mu$-open [3]. The set $c_\kappa$ (briefly $c_\kappa A$) is defined in [3] as the following:

$$c_\kappa(A) = \{x \in X : \kappa(M) \cap A \neq \emptyset \text{ for every } \mu\text{-open set } M \text{ containing } x\}.$$  

**Definition 3.1.** Let $(X, \mu)$ and $(Y, \nu)$ be generalized topological spaces. A function $f : (X, \mu) \to (Y, \nu)$ is said to be $(\kappa, \lambda)$-continuous if $x \in X$ and $N \in \nu$, $f(x) \in N$ imply the existence of $M \in \mu$ such that $x \in M$ and $f(\kappa M) \subseteq \lambda N$.

**Theorem 2.2** ([3]). Let $(X, \mu)$ and $(Y, \nu)$ be generalized topological spaces and $f : (X, \mu) \to (Y, \nu)$ a $(\kappa, \lambda)$-continuous function. Then the following hold:

1. $f(c_\mu(A)) \subset c_\lambda(f(A))$ holds for every subset $A$ of $(X, \mu)$.
2. for every $\lambda_\nu$-open set $B$ of $(Y, \nu)$, $f^{-1}(B)$ is $\kappa_\mu$-open in $(X, \mu)$.

3. Enlargement-separation axioms

**Definition 3.3.** Let $\kappa : \mu \to P(X)$ be an enlargement and $A$ a subset of $X$. Then the $\kappa_\mu$-closure of $A$ is denoted by $c_{\kappa_\mu}(A)$, and it is defined as the intersection of all $\kappa_\mu$-closed sets containing $A$.

**Remark 3.2.** Since the collection of all $\kappa_\mu$-open sets is a generalized topology on $X$, then for any $A \subseteq X$, $c_{\kappa_\mu}(A)$ is a $\kappa_\mu$-closed set.

**Proposition 3.3.** Let $\kappa : \mu \to P(X)$ be an enlargement and $A$ a subset of $X$. Then $c_{\kappa_\mu}(A) = \{y \in X : V \cap A \neq \emptyset \text{ for every } V \in \kappa_\mu \text{ such that } y \in V\}$. 

**Proof.** Denote $E = \{y \in X : V \cap A \neq \emptyset \text{ for every } V \in \kappa_\mu \text{ such that } y \in V\}$. We shall prove that $c_{\kappa_\mu}(A) = E$. Let $x \notin E$. Then there exists a $\kappa_\mu$-open set $V$ containing $x$ such that $V \cap A = \emptyset$. This implies that $X \setminus V$ is $\kappa_\mu$-closed and $A \subseteq X \setminus V$. Hence $c_{\kappa_\mu}(A) \subset X \setminus V$. It follows that $x \notin c_{\kappa_\mu}(A)$. Thus we have that $c_{\kappa_\mu}(A) \subset E$. Conversely, let $x \notin c_{\kappa_\mu}(A)$. Then there exists a $\kappa_\mu$-closed set $F$ such that $A \subseteq F$ and $x \notin F$. Then we have that $x \in X \setminus F$, $X \setminus F \in \kappa_\mu$ and $(X \setminus F) \cap A = \emptyset$. This implies that $x \notin E$. Hence $E \subset c_{\kappa_\mu}(A)$. Therefore $c_{\kappa_\mu}(A) = E$. \qed

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Example 3.4. Let $X = \{a, b, c, d\}$ and $\mu = P(X) \setminus \{\text{all proper subsets of } X \text{ which contains } d\}$. The enlargement $\kappa$ adds the element $d$ to each nonempty $\mu$-open set. Then $\kappa_{\mu} = \{\emptyset, X\}$. Now put $A = \{a\}$. Obviously $c_{\kappa_{\mu}}(A) = X$ and $c_{\kappa}(A) = \{a, d\}$. This example shows that $c_{\kappa} \not\subseteq c_{\kappa_{\mu}}$.

Example 3.5. Let $X = \mathbb{R}$ be the real line and $\mu = \{\emptyset, \mathbb{R}\} \cup \{\mathbb{R} \setminus \{x\}, x \neq 0\}$. The enlargement $\kappa$ is defined as $\kappa(A) = c_{\mu}(A)$. Then $\kappa_{\mu} = \{\emptyset, X\}$.

Example 3.6. Let $X = \mathbb{R}$ and $\mu = \{\emptyset, \mathbb{R}\} \cup \{A_a = (a, +\infty) \text{ for all } a \in \mathbb{R}\}$. The enlargement map $\kappa$ is defined as follows:

$$\kappa(A) = \begin{cases} A & \text{if } A = (0, +\infty), \\ \mathbb{R} & \text{if } A \neq (0, +\infty), \\ \emptyset & \text{if } A = \emptyset. \end{cases}$$

The generalized $\kappa_{\mu}$ topology on $X$ is $\{\emptyset, \mathbb{R}, (0, +\infty)\}$.

Definition 3.7. An enlargement $\kappa$ on $\mu$ is said to be open, if for every $\mu$-neighborhood $U$ of $x \in X$, there exists a $\kappa_{\mu}$-open set $B$ such that $x \in B$ and $\kappa(U) \supseteq B$.

Example 3.8. Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Define $\kappa : \mu \to P(X)$ as follows:

$$\kappa(A) = \begin{cases} A & \text{if } b \in A, \\ c_{\mu}(A) & \text{if } b \notin A. \end{cases}$$

The enlargement $\kappa$ on $\mu$ is open.

Proposition 3.9. If $\kappa : \mu \to P(X)$ is an open enlargement and $A$ a subset of $X$, then $c_{\kappa}(A) = c_{\kappa_{\mu}}(A)$ and $c_{\mu}(c_{\kappa}(A)) = c_{\kappa}(A)$ hold, and $c_{\kappa}(A)$ is $\kappa_{\mu}$-closed in $(X, \mu)$.

Proof. Suppose that $x \notin c_{\kappa}(A)$. Then there exists a $\mu$-open set $U$ containing $x$ such that $\kappa(U) \cap A = \emptyset$. Since $\kappa$ is an open enlargement, by Definition 3.7, there exists a $\kappa_{\mu}$-open set $V$ such that $x \in V \subseteq \kappa(U)$ and so $V \cap A = \emptyset$. By Proposition 3.3, $x \notin c_{\kappa_{\mu}}(A)$; it follows that $c_{\kappa_{\mu}}(A) \subseteq c_{\kappa}(A)$. By Corollary 1.7 of [3], we have $c_{\kappa}(A) \subseteq c_{\kappa_{\mu}}(A)$. In consequence, we obtain that $c_{\kappa}(c_{\kappa}(A)) = c_{\kappa}(A)$. By Proposition 1.3 of [3], we obtain that $c_{\kappa}(A)$ is a $\kappa_{\mu}$-closed in $(X, \mu)$.

Definition 3.10 ([6]). Let $\mu$ be a generalized topology on $X$ and $\kappa : \mu \to P(X)$ an enlargement of $\mu$. Then a subset $A$ of a generalized topological space $(X, \mu)$ is said to be a generalized $\kappa_{\mu}$-closed (abbreviated by $g.\kappa_{\mu}$-closed) set in $(X, \mu)$, if $c_{\kappa}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \kappa_{\mu}$.

Proposition 3.11. Every $\kappa_{\mu}$-closed set is $g.\kappa_{\mu}$-closed.

Proof. Straightforward.

Remark 3.12. A subset $A$ is $g.\text{id}_{\mu}$-closed if and only if $A$ is $g_{\mu}$-closed in the sense of Maragathamalli et. al. [5].

Theorem 3.13 ([6]). Let $\kappa$ be an enlargement of a generalized topological space $(X, \mu)$. If $A$ is $g.\kappa_{\mu}$-closed in $(X, \mu)$, then $c_{\kappa}(\{x\}) \cap A \neq \emptyset$ for every $x \in c_{\kappa}(A)$.
Proof. Let \( A \) be a \( g, \kappa, \mu \)-closed set of \((X, \mu)\). Suppose that there exists a point \( x \in c_\kappa(A) \) such that \( c_\kappa(\{x\}) \cap A = \emptyset \). By Proposition 1.3 of [3], \( c_\kappa(\{x\}) \) is \( \mu \)-closed. Put \( U = X \setminus c_\kappa(\{x\}) \). Then, we have that \( A \subset U \), \( x \notin U \) and \( U \) is a \( \mu \)-open set of \((X, \mu)\). Since \( A \) is a \( g, \kappa, \mu \)-closed set, \( c_\kappa(A) \subset U \). Thus, we have \( x \notin c_\kappa(A) \). This is a contradiction.

The converse of the above theorem is not necessarily true, as we can see.

**Example 3.14.** Let \( N \) be the set of all natural numbers and \( \mu \) the discrete topology on \( N \). Let \( i_0 \) be a fixed odd number. Define \( \kappa : \mu \rightarrow P(N) \) as follows:

\[
\kappa(\{n\}) = \begin{cases} 
\{2i : i \in N\} & \text{if } n \text{ is an even number}, \\
\{2i + 1 : i \in N\} & \text{if } n = i_0, \\
\{n\} & \text{if } n \text{ is an odd number } \neq i_0,
\end{cases}
\]

and \( \kappa(A) = N \) for the rest.

Clearly, \( \kappa \) is an enlargement on \( \mu \). Take \( A = \{2, 4\} \). It is easy to see that \( c_\kappa(A) = \{2i : i \in N\} \) and \( c_\kappa(\{x\}) \cap A \neq \emptyset \) for every \( x \in c_\kappa(A) \), but \( A \) is not a \( g, \kappa, \mu \)-closed set.

**Theorem 3.15.** Let \( \mu \) be a generalized topology on \( X \) and \( \kappa : \mu \rightarrow P(X) \) an enlargement on \( \mu \).

1. If a subset \( A \) is \( g, \kappa, \mu \)-closed in \((X, \mu)\), then \( c_\kappa(A) \setminus A \) does not contain any nonempty \( \kappa, \mu \)-closed set.

2. If \( \kappa : \mu \rightarrow P(X) \) is an open enlargement on \((X, \mu)\), then the converse of (1) is true.

**Remark 3.16.** The Theorem 4.1 of [6] is not true, because the condition that \( \kappa \) is an open enlargement can not be omitted, as we show in the following example.

**Example 3.17.** In the Example 3.14, \( \mu \) is not an open enlargement. If we take \( A = \{2, 4\} \), it is easy to see that \( c_\kappa(A) \setminus A \) does not contain any nonempty \( \kappa, \mu \)-closed set and \( A \) is not a \( g, \kappa, \mu \)-closed set.

**Lemma 3.18** ([6]). Let \( A \) be a subset of a generalized topological space \((X, \mu)\) and \( \kappa : \mu \rightarrow P(X) \) an enlargement on \((X, \mu)\). Then, for each \( x \in X \), \( \{x\} \) is \( \kappa, \mu \)-closed or \( (X \setminus \{x\}) \) is a \( g, \kappa, \mu \)-closed set of \((X, \mu)\).

**Proof.** Suppose that \( \{x\} \) is not \( \kappa, \mu \)-closed. Then \( X \setminus \{x\} \) is not \( \kappa, \mu \)-open. Let \( U \) be any \( \kappa, \mu \)-open set such that \( X \setminus \{x\} \subset U \). Then, since \( U \supseteq X, c_\kappa(X \setminus \{x\}) \subset U \). Therefore, \( X \setminus \{x\} \) is \( g, \kappa, \mu \)-closed.

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Definition 3.19. A generalized topological space \((X, \mu)\) is said to be a \(\kappa\)-\(\text{T}_{1/2}\) space, if every \(g, \kappa\)-\(\mu\)-closed set of \((X, \mu)\) is \(\kappa\)-\(\mu\)-closed.

Theorem 3.20. A generalized topological space \((X, \mu)\) is \(\kappa\)-\(\text{T}_{1/2}\) if and only if, for each \(x \in X\), \(\{x\}\) is \(\kappa\)-\(\mu\)-closed or \(\kappa\)-\(\mu\)-open in \((X, \mu)\).

\[
\text{Proof.}
\]

Necessity: It is obtained by Lemma 3.18 and Definition 3.19. Sufficiency: Let \(F\) be \(g, \kappa\)-\(\mu\)-closed in \((X, \mu)\). We shall prove that \(c_{\kappa, \mu}(F) = F\). It is sufficient to show that \(c_{\kappa, \mu}(F) \subset F\). Assume that there exists a point \(x\) such that \(x \in c_{\kappa, \mu}(F) \setminus F\). Then, by assumption, \(\{x\}\) is \(\kappa\)-\(\mu\)-closed or \(\kappa\)-\(\mu\)-open.

Case(i): \(\{x\}\) is \(\kappa\)-\(\mu\)-closed set. For this case, we have a \(\kappa\)-\(\mu\)-open set \(\{x\}\) such that \(\{x\} \subset c_{\kappa, \mu}(F) \setminus F\). This is a contradiction to Theorem 3.15 (1).

Case(ii): \(\{x\}\) is \(\kappa\)-\(\mu\)-open set. Using Corollary 1.7 of [3], we have \(x \in c_{\kappa, \mu}(F)\). Since \(\{x\}\) is \(\kappa\)-\(\mu\)-open, it implies that \(\{x\} \cap F \neq \emptyset\). This is a contradiction. Thus, we have that \(c_{\kappa}(F) = F\), and so, by Proposition 1.4 of [3], \(F\) is \(\kappa\)-\(\mu\)-closed. 

Definition 3.21. Let \(\kappa : \mu \to P(X)\) be an enlargement. A generalized topological space \((X, \mu)\) is said to be:

1. \(\kappa\)-\(\text{T}_0\) if for any two distinct points \(x, y \in X\) there exists a \(\mu\)-open set \(U\) such that either \(x \in U\) and \(y \notin \kappa(U)\) or \(y \in U\) and \(x \notin \kappa(U)\).

2. \(\kappa\)-\(\text{T}_1\) if for any two distinct points \(x, y \in X\) there exist two \(\mu\)-open sets \(U\) and \(V\) containing \(x\) and \(y\), respectively such that \(y \notin \kappa(U)\) and \(x \notin \kappa(V)\).

3. \(\kappa\)-\(\text{T}_2\) if for any two distinct points \(x, y \in X\) there exist two \(\mu\)-open sets \(U\) and \(V\) containing \(x\) and \(y\), respectively such that \(\kappa(U) \cap \kappa(V) = \emptyset\).

Theorem 3.22. Let \(A\) be a subset of a generalized topological space \((X, \mu)\) and \(\kappa : \mu \to P(X)\) an open enlargement on \((X, \mu)\). Then \((X, \mu)\) is a \(\kappa\)-\(\text{T}_0\) space if and only if for each pair \(x, y \in X\) with \(x \neq y\), \(c_{\kappa}(\{x\}) = c_{\kappa}(\{y\})\) holds.

\[
\text{Proof.}
\]

Let \(x\) and \(y\) be any two distinct points of a \(\kappa\)-\(\text{T}_0\) space. Then, by Definition 3.21, there exists a \(\mu\)-open set \(U\) such that \(x \in U\) and \(y \notin \kappa(U)\). It follows that there exists a \(\mu\)-open set \(S\) such that \(x \in S\) and \(S \subset \kappa(U)\). Hence, \(y \in X \setminus \kappa(U) \subset X \setminus S\). Because \(X \setminus S\) is a \(\mu\)-closed set, we obtain that \(c_{\kappa}(\{y\}) \subset X \setminus S\), and so \(c_{\kappa}(\{x\}) \neq c_{\kappa}(\{y\})\). Conversely, suppose that \(x \neq y\) for any \(x, y \in X\). Then, we have that \(c_{\kappa}(\{x\}) \neq c_{\kappa}(\{y\})\). Thus, we assume that there exists \(z \in c_{\kappa}(\{x\})\) but \(z \notin c_{\kappa}(\{y\})\). If \(x \in c_{\kappa}(\{y\})\), then we obtain \(c_{\kappa}(\{x\}) \subset c_{\kappa}(\{y\})\). This implies that \(z \in c_{\kappa}(\{y\})\). This is a contradiction; in consequence, \(x \in c_{\kappa}(\{y\})\). Therefore, there exists a \(\mu\)-open set \(W\) such that \(x \in W\) and \(\kappa(W) \cap \{y\} = \emptyset\). Thus, we have that \(x \in W\) and \(y \notin \kappa(W)\). Hence, \((X, \mu)\) is a \(\kappa\)-\(\text{T}_0\) space.

Example 3.23. In the Example 3.14, take \(A = \{2, 4\}; \text{ then } c_{\kappa}(A) - A = \{2i : i \in N - \{1, 2\}\} \text{ does not contain any nonempty } \kappa\)-\(\mu\)-open set, and \(A\) is not a \(g, \kappa\)-\(\mu\)-closed set.

Theorem 3.24. A generalized topological space \((X, \mu)\) is \(\kappa\)-\(\text{T}_1\) if and only if every singleton set of \(X\) is \(\kappa\)-\(\mu\)-closed.
Let \( \kappa \) be a generalised topology. Then the sequence \( \{x_k\} \) is said to be \( \kappa \)-convergent to a point \( x_0 \in X \), denoted \( x_k \xrightarrow{\kappa} x_0 \), if for every \( \mu \)-open set \( U \) containing \( x_0 \) there exists a positive integer \( n \) such that \( x_k \in \kappa(U) \) for all \( k \geq n \).

**Definition 3.25.** Let \((X, \mu)\) be a generalised topological space. Then the sequence \( \{x_k\} \) is said to be \( \kappa \)-convergent to a point \( x_0 \in X \), denoted \( x_k \xrightarrow{\kappa} x_0 \), if for every \( \mu \)-open set \( U \) containing \( x_0 \) there exists a positive integer \( n \) such that \( x_k \in \kappa(U) \) for all \( k \geq n \).

**Theorem 3.26.** Let \((X, \mu)\) be a \( \kappa \)-\( T_2 \) space. If \( \{x_k\} \) is a \( \kappa \)-convergent sequence, then it \( \kappa \)-converges to at most one point.

**Proof.** Let \( \{x_k\} \) be a sequence in \( X \) \( \kappa \) converging to \( x \) and \( y \). Then by definition of \( \kappa \)-\( T_2 \) space, there exist \( U, V \in \mu \) such that \( x \in U, y \in V \) and \( \kappa(U) \cap \kappa(V) = \emptyset \). Since \( x_k \xrightarrow{\kappa} x \), there exists a positive integer \( n_1 \) such that \( x_k \in \kappa(U) \) for all \( k \geq n_1 \). Also \( x_k \xrightarrow{\kappa} y \), therefore there exists a positive integer \( n_2 \) such that \( x_k \in \kappa(V) \), for all \( k \geq n_2 \). Let \( n_0 = \max (n_1, n_2) \). Then \( x_k \in \kappa(U) \) and \( x_k \in \kappa(V) \), for all \( k \geq n_0 \) or \( x_k \in \kappa(U) \cap \kappa(V) \), for all \( k \geq n_0 \). This contradiction proves that \( \{x_k\} \) \( \kappa \)-converges to at most one point.

**Remark 3.27.** Note that the above results generalize the well known separation axioms in general topology in an structure more weaker than a topology.

### 4. Additional Properties

**Proposition 4.1.** Let \( f : (X, \mu) \to (Y, \nu) \) be a \((\kappa, \lambda)\)-continuous injection. If \((Y, \nu)\) is \( \lambda \)-\( T_1 \) (resp. \( \lambda \)-\( T_2 \)), then \((X, \mu)\) is \( \kappa \)-\( T_1 \) (resp. \( \kappa \)-\( T_2 \)).

**Proof.** Suppose that \((Y, \nu)\) is \( \lambda \)-\( T_2 \). Let \( x \) and \( x' \) be distinct points of \( X \). Then there exist two open sets \( V \) and \( W \) of \( Y \) such that \( f(x) \in V, f(x') \in W \) and \( \lambda(V) \cap \lambda(W) = \emptyset \). Since \( f \) is \((\kappa, \lambda)\)-continuous, for \( V \) and \( W \) there exist two open sets \( U, S \) such that \( x \in U, x' \in S, f(\kappa(U)) \subset \lambda(V) \) and \( f(\kappa(S)) \subset \lambda(W) \). Therefore, we have \( \kappa(U) \cap \kappa(S) = \emptyset \), and hence \((X, \mu)\) is \( \kappa \)-\( T_2 \). The proof of the case of \( \lambda \)-\( T_1 \) is similar.

In [4] the notion of product of generalised topologies is defined. Let \( \mu \) and \( \nu \) be two generalised topologies, and \( \beta \) the collection of all sets \( U \times V \), where \( U \in \mu \) and \( V \in \nu \). Clearly \( \emptyset \in \beta \), so we can define a generalised topology \( \mu \times \nu = \mu \times \nu(\beta) \) having \( \beta \) for base. We call \( \mu \times \nu \) the product of the generalised topologies \( \mu \) and \( \nu \).

**Definition 4.2.** An enlargement \( \kappa : \mu \times \nu \to \mathcal{P}(X \times Y) \) is said to be associated with \( \kappa_1 \) and \( \kappa_2 \), if \( \kappa(U \times V) = \kappa_1(U) \times \kappa_2(V) \) holds for each \((\neq \emptyset) U \in \mu, (\neq \emptyset) V \in \nu \).

**Definition 4.3.** An enlargement \( \kappa : \mu \times \nu \to \mathcal{P}(X \times X) \) is said to be regular with respect to \( \kappa_1 \) and \( \kappa_2 \), if for each point \((x, y) \in X \times Y \) and each \( \mu \times \nu \)-open set \( W \) containing \((x, y) \), there exists \( U \in \mu \) and \( V \in \nu \) such that \( x \in U, y \in V \) and \( \kappa_1(U) \times \kappa_2(V) \subset \kappa(W) \).

**Proposition 4.4.** Let \( \kappa : \mu \times \nu \to \mathcal{P}(X \times X) \) be an enlargement associated with \( \kappa_1 \) and \( \kappa_2 \). If \( f : (X, \mu) \to (Y, \nu) \) is \((\kappa_1, \kappa_2)\)-continuous and \((Y, \nu)\) is a \( \kappa_2 \)-\( T_2 \) space, then the set \( A = \{(x, y) \in X \times X : f(x) = f(y)\} \) is a \( \kappa \)-closed set of \((X \times X, \mu \times \mu)\).
Proof. We show that \( c_\kappa(A) \subset A \). Let \((x, y) \in X \times X \setminus A\). Then, there exist \(U, V \in \nu\) such that \(f(x) \in U, f(y) \in V\) and \(\kappa_2(U) \cap \kappa_2(V) = \emptyset\). Moreover, for \(U\) and \(V\) there exist \(W, S \in \mu\) such that \(x \in W, y \in S, f(\kappa_1(W)) \subset \kappa_2(U)\) and \(f(\kappa_1(S)) \subset \kappa_2(V)\). Therefore, we have \(\kappa(W \times S) \cap A = \emptyset\). This shows that \((x, y) \notin c_\kappa(A)\). 

Corollary 4.5. If \(\kappa : \mu \times \mu \to \mathcal{P}(X \times X)\) is an enlargement associated with \(\kappa_1\) and \(\kappa_2\) and it is regular with respect to \(\kappa_1\) and \(\kappa_2\). A generalized topological space \((X, \mu)\) is \(\kappa_1\)-T2 if and only if the diagonal set \(\Delta = \{(x, x) : x \in X\}\) is \(\kappa\)-closed in \((X \times X, \mu \times \mu)\).

Proposition 4.6. Let \(\kappa : \mu \times \nu \to \mathcal{P}(X \times Y)\) be an enlargement associated with \(\kappa_1\) and \(\kappa_2\). If \(f : (X, \mu) \to (Y, \nu)\) is \((\kappa_1, \kappa_2)\)-continuous and \((Y, \nu)\) is a \(\kappa_2\)-T2 space, then the graph of \(f, \mathcal{G}(f) = \{(x, f(x)) \in X \times Y\}\) is a \(\kappa\)-closed set of \((X \times Y, \mu \times \nu)\).

**Proof.** The proof is similar to that of Proposition 4.4.

Definition 4.7. An enlargement \(\kappa\) on \(\mu\) is said to be regular, if for any \(\mu\)-open neighborhoods \(U, V\) of \(x \in X\), there exists a \(\mu\)-open neighborhood \(W\) of \(x\) such that \(\kappa(U) \cap \kappa(W) = \kappa(W)\).

Theorem 4.8. Suppose that \(\kappa_1\) is a regular enlargement and \(\kappa : \mu \times \nu \to \mathcal{P}(X \times Y)\) is regular with respect to \(\kappa_1\) and \(\kappa_2\). Let \(f : (X, \mu) \to (Y, \nu)\) be a function whose graph \(\mathcal{G}(f)\) is \(\kappa\)-closed in \((X \times Y, \mu \times \nu)\). If a subset \(B\) is \(\kappa_2\)-compact in \((Y, \nu)\), then \(f^{-1}(B)\) is \(\kappa_1\)-closed in \((X, \mu)\).

**Proof.** Suppose that \(f^{-1}(B)\) is not \(\kappa_1\)-closed. Then, there exists a point \(x\) such that \(x \in c_{\kappa_1}(f^{-1}(B))\) and \(x \notin f^{-1}(B)\). Since \((x, b) \notin \mathcal{G}(f)\) for each \(b \in B\) and \(\mathcal{G}(f) \supseteq c_{\kappa_1}(\mathcal{G}(f))\), there exists a \(\mu \times \nu\)-open set \(W\) such that \((x, b) \in W\) and \(\kappa(W) \cap \mathcal{G}(f) = \emptyset\). By the regularity of \(\kappa\), for each \(b \in B\) we can take two sets \(U(b) \in \mu\) and \(V(b) \in \nu\) such that \(x \in U(b), b \in V(b)\) and \(\kappa_1(U(b)) \times \kappa_2(V(b)) \subset \kappa(W)\). Then we have \(f(\kappa_1(U(b))) \cap \kappa_2(V(b)) = \emptyset\). Since \(\{V(b) : b \in B\}\) is a \(\nu\)-open cover of \(B\), there exists a finite number of points \(b_1, ..., b_n \in B\) such that \(B \subset \bigcup_{i=1}^{n} \kappa_2(V(b_i))\), by the \(\kappa_2\)-compactness of \(B\). By the regularity of \(\kappa_1\), there exists \(U \in \mu\) such that \(x \in U, \kappa_1(U) \subset \bigcap_{i=1}^{n} \kappa_1(U(b_i))\). Therefore, we have \(\kappa_1(U) \cap f^{-1}(B) \subset \bigcup_{i=1}^{n} \kappa_1(U(b_i)) \cap f^{-1}(\kappa_2(V(b_i))) = \emptyset\). This shows that \(x \notin c_{\kappa_1}(f^{-1}(B))\), thus we have a contradiction.

Theorem 4.9. Let \(f : (X, \mu) \to (Y, \nu)\) be a function whose graph \(\mathcal{G}(f)\) is \(\kappa\)-closed in \((X \times Y, \mu \times \nu)\), and suppose that the following conditions hold:

1. \(\kappa_1 : \mu \to \mathcal{P}(X)\) is open,
2. \(\kappa_2 : \nu \to \mathcal{P}(Y)\) is regular, and
3. \(\kappa : \mu \times \nu \to \mathcal{P}(X \times Y)\) is an enlargement associated with \(\kappa_1\) and \(\kappa_2\), and \(\kappa\) is regular with respect to \(\kappa_1\) and \(\kappa_2\).

If every cover of \(A\) by \(\kappa_1\)-open sets of \((X, \mu)\) has a finite subcover, then \(f(A)\) is \(\kappa_2\)-closed in \((Y, \nu)\).

**Proof.** The proof is similar to that of Theorem 4.8.

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Proposition 4.10. Let \( \kappa : \mu \times \nu \rightarrow P(X \times Y) \) be an enlargement associated with \( \kappa_1 \) and \( \kappa_2 \). If \( f : (X, \mu) \rightarrow (Y, \nu) \) is \((\kappa_1,\kappa_2)\)-continuous and \((Y, \nu)\) is a \( \kappa_2\)-\( T_2 \), then the graph of \( f \), \( G(f) = \{(x, f(x)) \in X \times Y \} \) is a \( \kappa_{\mu \times \nu} \)-closed set of \((X \times Y, \mu \times \nu)\).

Proof. The proof is similar to that of Proposition 4.4.

Definition 4.11. A function \( f : (X, \mu) \rightarrow (Y, \nu) \) is said to be \((\kappa, \lambda)\)-closed, if for any \( \kappa_{\mu}\)-closed set \( A \) of \((X, \mu)\), \( f(A) \) is \( \lambda_{\nu} \)-closed in \((Y, \nu)\).

Theorem 4.12. Suppose that \( f \) is \((\kappa, \lambda)\)-continuous and \((id, \lambda)\)-closed. If for every \( g, \kappa_{\mu}\)-closed set \( A \) of \((X, \mu)\), then the image \( f(A) \) is \( g, \lambda_{\nu} \)-closed.

Proof. Let \( V \) be any \( \lambda_{\nu} \)-open set of \((Y, \nu)\) such that \( f(A) \subset V \). By the Theorem 2.2 (2), \( f^{-1}(V) \) is \( \kappa_{\mu} \)-open. Since \( A \) is \( g, \kappa_{\mu} \)-closed and \( A \subset f^{-1}(V) \), we have \( c_{\kappa}(A) \subset f^{-1}(V) \), and hence \( f(c_{\kappa}(A)) \subset V \). It follows from Proposition 1.3 of [3] and our assumption that \( f(c_{\kappa}(A)) \) is \( \lambda_{\nu} \)-closed. Therefore we have \( c_{\lambda}(f(A)) \subset c_{\lambda}(f(c_{\kappa}(A))) = f(c_{\kappa}(A)) \subset V \). This implies \( f(A) \) is \( g, \lambda_{\nu} \)-closed.

Theorem 4.13. If \( f : (X, \mu) \rightarrow (Y, \nu) \) is \((\kappa, \lambda)\)-continuous and \((id, \lambda)\)-closed, if \( f \) is injective and \((Y, \nu)\) is \( \lambda\)-\( T_{1/2} \), then \((X, \mu)\) is \( \kappa\)-\( T_{1/2} \).

Proof. Let \( A \) be a \( g, \kappa_{\mu} \)-closed set of \((X, \mu)\). We show that \( A \) is \( \kappa_{\mu} \)-closed. By Theorem 4.12 and our assumptions it is obtained that \( f(A) \) is \( g, \lambda_{\nu} \)-closed, and hence \( f(A) \) is \( \lambda_{\nu} \)-closed. Since \( f \) is \((\kappa, \lambda)\)-continuous, \( f^{-1}(f(A)) \) is \( \kappa_{\mu} \)-closed by the using Theorem 2.2 (2).

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References


