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The Cauchy Problem for the Combustion Model in a Porous Medium with two Layers

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Overview:

- The model
- Monotone Iterative Method
- Fundamental solution
- Cauchy problem for the full system

The model

J.C. da Mota and S. Schecter (Jr. Dyn. Diff. Eq., **18** *(3), (2006))*:

$$(*) \qquad \begin{cases} \big((a_1+b_1y)u\big)_t + (c_1u)_x &= d_1 f(u,y) - q(u-v) + \lambda_1 u_{xx} \\ y_t &= -A_1 f(u,y) \\ \big((a_2+b_2z)v\big)_t + (c_2v)_x &= d_2 f(v,z) - q(v-u) + \lambda_2 v_{xx} \\ z_t &= -A_2 f(v,z) \end{cases}$$

$$x \in \mathbb{R}, \quad t > 0$$

$u = u(x,t),\ y = y(x,t)$ are the temperature and fuel concentration in one layer and $v = v(x,t),\ z = z(x,t)$ are the temperature and fuel concentration in the other layer;
f, g are given by

$$f(u,y) = y\, e^{-\frac{\tilde{E}}{u}}, \qquad f(v,z) = z\, e^{-\frac{\tilde{E}}{v}}$$

(reaction rate funcions / Arrhenius's law);

$a_i,\ b_i,\ c_i,\ d_i,\ A_i,\ \lambda_i,\ \ i = 1,2,\ \tilde{E}$ and q are nonnegative parameters;

q is the heat transfer coefficient.

Rewriting (∗):

Let $u_1 = u,\ y_1 = y,\ \ u_2 = v,\ y_2 = z,$ and apply product rule. Then (∗) writes

$$\begin{cases} (u_i)_t - \dfrac{\lambda_i}{a_i + b_i y_i}(u_i)_{xx} + \dfrac{c_i}{a_i + b_i y_i}(u_i)_x \\ \qquad\qquad\qquad = \dfrac{b_i A_i u_i + d_i}{a_i + b_i y_i} y_i f(u_i) + (-1)^i q \dfrac{u_1 - u_2}{a_i + b_i y_i}, \\ (y_i)_t = -A_i y_i f(u_i), \end{cases}$$

or

$$\begin{cases} (u_i)_t - L_i u_i = f_i(u_1, u_2, y_i) \\ (y_i)_t = -A_i y_i f(u_i) \end{cases}$$

where $i = 1, 2,$

$$L_i u = \frac{\lambda_i}{a_i + b_i y_i} u_{xx} - \frac{c_i}{a_i + b_i y_i} u_x,$$

$$f_i(u_1, u_2, y_i) = \frac{1}{a_i + b_i y_i}\big((A_i b_i u_i + d_i) f(u_i, y_i) + (-1)^i q(u_1 - u_2)\big),$$

Writing the full system $(*)$ as a reaction-diffusion system

Denoting $W = (u_1, y_1, u_2, y_2)$, the system $(*)$ can be written as

$$W_t = DW_{xx} + MW_x + F(W)$$

where

$$D = \begin{pmatrix} \frac{\lambda_1}{a_1+b_1y_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda_2}{a_2+b_2y_2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} -\frac{c_1}{a_1+b_1y_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{c_2}{a_2+b_2y_2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$F(W) = \begin{pmatrix} (b_1A_1u_1f(u_1, y_1) + d_1f(u_1, y_1) - q(u_1 - u2))/(a_1 + b_1y_1) \\ -A_1f(u_1, y_1) \\ (b_2A_2u_2f(u_2, y_2) + d_2f(u_2, y_2) + q(u_1 - u_2))/(a_2 + b_2y_2) \\ -A_2f(u_2, y_2) \end{pmatrix}.$$

Notice that the diffusion matrix D is not strictly positive.

Invariant regions

Using the Chueh-Conley-Smoller approach*, we can easily show that the set

$$\Sigma = \{(u, y, v, z)\,;\, 0 \le u,\ 0 \le y \le 1,\ 0 \le v,\ 0 \le z \le 1\,\}$$

is an invariant region for the system $(*)$. More precisely:

Theorem. *If $W = (u_1, y_1, u_2, y_2)$ is a smooth solution for $(*)$, in the domain $x \in \mathbb{R}, t > 0$, with an initial condition $W(x, 0) = W_0(x)$such that $W_0(x) \in \Sigma$ for all $x \in \mathbb{R}$ and, if there are positive constants c and ϵ such that $W(x, t) \in \Sigma$ for all (x, t) such that $|x| > c$ and $0 < t < \epsilon$, then $W(x, t) \in \Sigma$ for all $(x, t) \in \mathbb{R} \times [0, \infty)$.*

* K. N. Chueh, C. C. Conley, J. A. Smoller, *Positively invariant regions for systems of nonlinear diffusion equations*, Indiana Univ. Math. J. **26** (2) (1977), 373–392.

or Ch.14 in

J. A. Smoller, *Shock waves and reaction-diffusion equations*, Second edition, Springer-Verlag, New York (1994).

Reduced system:

Considering the concentrations functions y_1, y_2 as known, we obtain the system for the unknowns u_1 , u_2:

$$\left\{ \; (u_i)_t - L_i(u_i) = f_i(x,t,u_1,u_2) \right.$$

where

$$\begin{aligned} L_i u &= \frac{1}{a_i + b_i y_i(x,t)}\big(\lambda_1 u_{xx} - c_i u_x\big), \\ f_i(x,t,u_1,u_2) &= \frac{1}{a_i + b_i y_i(x,t)}\big((A_i b_i u_i + d_i) f(u_i, y_i) + (-1)^i q(u_1 - u_2)\big). \end{aligned} \tag{1}$$

The reaction function $F = (f_1, f_2)$ is quasi-monotone nondecreasing:

$$\frac{\partial f_1}{\partial u_2} = \frac{q}{a_1 + b_1 y} \geq 0$$

$$\frac{\partial f_2}{\partial u_1} = \frac{q}{a_2 + b_2 z} \geq 0.$$

F is also Lipschitz continuous:

$$|f_1(U) - f_1(\tilde{U})| \leq c_1 \|y_1\|_\infty |u_1 - \tilde{u}_2| + \frac{q}{a_1}|U - \tilde{U}|,$$

$$|f_2(U) - f_2(\tilde{U})| \leq c_2 \|y_2\|_\infty |u_2 - \tilde{u}_2| + \frac{q}{a_2}|U - \tilde{U}|.$$

Besides,

$$\hat{U} = (0,0) \;\text{ is a lower solution.}$$

and

$$\tilde{U} = (\varphi(t), \varphi(t))$$

is an upper solution, where

$$\varphi(t) = \big(\parallel u_0 \parallel_\infty + \parallel v_0 \parallel_\infty + \beta \big) e^{\alpha t} - \beta$$

and

$$\alpha = \max\big\{A_1 b_1 / a_1,\, A_2 b_2 / a_2\big\}, \quad \text{and} \;\; \beta = \max\big\{d_1 / A_1 b_1,\, d_2 / A_2 b_2\big\}\,.$$

The Cauchy problem for the reduced system/Monotone iterative method:

Write $U = (u_1, u_2)$, $\mathbb{L}U = U_t - (L_1 u,\, L_2 v)$ and consider the Cauchy problem

$$\begin{cases} \mathbb{L}U = F(x,t,U), & x \in \Omega_T \\ U(x,0) = \big(u_0(x), v_0(x)\big), & x \in \mathbb{R}, \end{cases}$$

$\Omega_T := \mathbb{R} \times (0,T]$ $(T > 0)$.

The Monotone Iterative Method[1]

Define the sequence of functions $U^{(k)} = (u^{(k)}, v^{(k)})$ by

$$\begin{cases} \mathbb{L}_K U^{(k)} = F_K(x,t,U^{(k-1)}) & \text{in } \Omega_T \\ U^{(k)}(x,0) = (u_0(x), v_0(x)) & \text{in } \mathbb{R}^n\,, \end{cases}$$

where $\mathbb{L}_K = \mathbb{L} + K$ and $F_K \equiv KU + F$ and K is the Lipschitz constant of F.

$U^{(k)}$ is denoted by $\overline{U}^{(k)}$ if the initial iteration $U^{(0)}$ is given by $U^{(0)} = \tilde{U}$ and by $\underline{U}^{(k)}$ if the initial iteration $U^{(0)}$ is given by $U^{(0)} = \hat{U}$.

Monotone property:

$$\hat{U} \le \underline{U}^{(k)} \le \underline{U}^{(k+1)} \le \overline{U}^{(k+1)} \le \overline{U}^{(k)} \le \tilde{U}$$

The inequalities are understood in the componentwise sense, i.e. if $U_1 = (u_1, v_1)$ and $U_2 = (u_2, v_2)$ then $U_1 \le U_2$ means $u_1 \le u_2$ and $v_1 \le v_2$. Consequently, there exist the pointwise limits

$$\overline{U}(x,t) \equiv \lim_{k\to\infty} \overline{U}^{(k)}(x,t) \quad \text{and} \quad \underline{U}(x,t) \equiv \lim_{k\to\infty} \underline{U}^{(k)}(x,t)$$

for each $(x,t) \in \overline{\Omega_T}$.

[1] C.V. Pao: *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York and London, (1992);
Parabolic systems in unbounded domains I. Existence and dynamics, Jr. Math. An. Appl. **217** (1998).

The monotone property is obtained using the "Maximum principle" and the exponential growth at infinity:

$$|\tilde{U}(x,t)| \leq A_0 \exp(b|x|^2), \quad |x| >> 1,$$

where A_0 and b are positive constants independent of $|x| >> 1$ and $t \in [0,T]$.

i.e. Phragman-Lindelöf principle*:

Let $\mathcal{L}$ be a parabolic parabolic operator,

$$\mathcal{L} = \partial_t - \sum_{j,k=1}^{n} a_{jk}\partial_{x_j x_k} + \sum_{j=1}^{n} b_j \partial_{x_j} + c.$$

If $\mathcal{L}w \geq 0$ in $\mathbb{R}^n \times (0,T)$, $w(x,0) \geq 0$ in $\mathbb{R}^n$ and there exists a $\delta > 0$ such that

$$\limsup_{R\to\infty} \left[\mathrm{e}^{-\delta R^2} \min\{w(x,t); 0 \leq t \leq T, \ |x| = R\} \right]$$

then $w(x,t) \geq 0$ in $\mathbb{R}^n \times (0,T)$.

* e.g. Protter, M. H. and Weinberger, H. F., *Maximum principles in differential equation*, Springer-Verlang (1984).

Theorem. *Let u_0 and v_0 be nonnegative continuous and bounded functions defined on $\mathbb{R}$. Then, given any $T > 0$ and nonnegative and bounded functions y, z in $C(\overline{\Omega_T}) \cap C^{0,1}(\Omega_T)$, where $\Omega_T \equiv \mathbb{R} \times (0, T]$, the sequences $\overline{U}^{(k)}$, $\underline{U}^{(k)}$ converge to the unique solution $U \equiv (u, v)$ of the above Cauchy problem in $C(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ satisfying the exponential growth at infinity and*

$$0 \leq u,\ v \leq \varphi$$

in $\overline{\Omega_T}$, where φ is the upper solution defined above.

The Cauchy problem for the full system

The Cauchy problem:

$$\left\{\begin{array}{l} (u_i)_t - \dfrac{\lambda_i}{a_i + b_i y_i}(u_i)_{xx} + \dfrac{c_i}{a_i + b_i y_i}(u_i)_x = \\ \qquad = \dfrac{b_i A_i u_i + d_i}{a_i + b_i y_i} y_i f(u_i) + (-1)^i q \dfrac{u_1 - u_2}{a_i + b_i y_i}, \\ (y_i)_t = -A_i y_i f(u_i), \\ (u_i(x,0), y_i(x,0)) = (u_{i,0}(x), y_{i,0}(x)), \end{array}\right. \tag{2}$$

where $u_{i,0}$ and $y_{i,0}$ are given functions, $t > 0$, $x \in \mathbb{R}$, $a_i, b_i, c_i, d_i, A_i, \lambda_i, E$, $i = 1, 2$, are nonnegative constants and

$$f(u) = \left\{\begin{array}{l} e^{-\frac{E}{u}},\ se\, u > 0 \\ 0,\ se\, u \leq 0 \end{array}\right. . \tag{3}$$

We show existence of a local solution (local in time) and global solution if the initial temperatures are in L^p, for some $1 < p < \infty$.

Our proof is strongly based on fundamental solution properties.

The system can be seen in the variables $u_1,\ u_2$ only. In fact, substituting

$$y_i(x,t) = y_{i,0}(x)\mathrm{e}^{-A_i \int_0^t f(u_i(x,s))ds}$$

in the first equation, equation for u_i, we see that the system is of the following form:

$$\left\{ \begin{array}{r} (u_i)_t - a(x, \int_0^t f(u_i(x,s))ds)\ (u_i)_{xx} + b(x, \int_0^t f(u_i(x,s))ds)\ (u_i)_x \\ = F_i(x, u_1, u_2, \int_0^t f(u_i(x,s))ds) \end{array} \right.$$

Fundamental solution:

Consider the equation

$$Lu \equiv \frac{\partial u}{\partial t} - a(x,t)\frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial u}{\partial x} + c(x,t)u = 0\,, \tag{4}$$

where a, b and c are functions defined in $\Omega_T = \{(x,t);\, x \in \mathbb{R},\, 0 \leq t \leq T\}$.

We assume that

(A1) L is uniformly parabolic in Ω_T, i.e., there are positive constants λ_0 and λ_1 such that

$$\lambda_0 \leq a(x,t) \leq \lambda_1, \quad \text{for all } (x,t) \in \Omega_T, \tag{5}$$

(A2) The coefficients of L are bounded and Hölder continuous functions in Ω_T.

Definition 1 *A fundamental solution of $Lu = 0$ is a function $\Gamma(x,t,\xi,\tau)$, defined for all $(x,t) \in \Omega_T$, $(\xi,\tau) \in \Omega_T$, $t > \tau$, which satisfies the following conditions:*

1. *$L\Gamma = 0$ as a function of (x,t), for each fixed $(\xi,\tau) \in \Omega_T$.*
2. *For all real and continuous function $f(x)$, if $|f(x)| \leq Ke^{hx^2}$, $h \leq \frac{1}{4\lambda_1 T}$, then*

$$\lim_{t \to \tau} \int_{\mathbb{R}} \Gamma(x,t,\xi,\tau) f(\xi) d\xi = f(x).$$

The method, called *parametrix method*, to prove existence of the fundamental solution of (4) is due to E. E. Levi (*Sulle equazioni totalmente ellitche ale derivate parziale*, Rend. del Circ. Mat. Palermo, **24**, (1907), 275–317.)

The method is detailed in the books:

A. Friedman *Partial differential equations of parabolic type*, Dover Publications, New York, (2008).

O. A. Ladyzenskaja, V. A. Solonnikov e N. N. Ural'ceva *Linear and quasilinear equations of parabolic type*.

The fundamental solution is given by

$$\Gamma(x,t,\xi,\tau) = Z(x-\xi,\xi,t,\tau) + \int_\tau^t \int_{\mathbb{R}} Z(x-y,y,t,\sigma)\phi(y,\xi,\sigma,\tau)dyd\sigma\,, \quad (6)$$

where for (x,t), $(\xi,\tau) \in \Omega_T$, $\tau < t$,

$$Z(x,t,\xi,\tau) = \frac{1}{(4\pi a(\xi,\tau)(t-\tau))^{\frac{1}{2}}} e^{-\frac{(x-\xi)^2}{4a(\xi,\tau)(t-\tau)}}, \quad (7)$$

$$\phi(x,t,\xi,\tau) = \sum_{m=1}^{\infty} (-1)^m (LZ)_m(x,t,\xi,\tau), \quad (8)$$

with

$$(LZ)_{m+1}(x,t,\xi,\tau) = \int_\tau^t \int_{\mathbb{R}} [LZ(x,t,y,\sigma)](LZ)_m(y,\sigma,\xi,\tau)dyd\sigma, \quad (9)$$

where $(LZ)_1 = LZ = (a(\xi,\tau) - a(x,t))\frac{\partial^2 Z}{\partial x^2} + b\frac{\partial Z}{\partial x} + cZ$.

Representation formula

Consider the Cauchy problem

$$\begin{cases} Lu(x,t) = f(x,t), & em \quad \mathbb{R} \times (0,T], \\ u(x,0) = u_0(x), & em \quad \mathbb{R}, \end{cases} \tag{10}$$

where L is defined as in (4) with Höder continuous coefficients and $|f(x,t)|, |u_0(x)| \leq Ke^{hx^2}$, where K and h are positive constants, with $h < \frac{1}{4\lambda_1 T}$.

The following theorem gives a representation formula for the solution of the Cauchy Problem (10) using the fundamental solution.

Theorem 1 *If $f(x,t)$ and $u_0(x)$ are continuous functions in Ω_T and $\mathbb{R}$, respectively, and, furthermore, $f(x,t)$ is locally Hölder continuous in x (exponent α), uniformly with respect to t, then the function*

$$u(x,t) = \int_{\mathbb{R}} \Gamma(x,\xi,t,0)u_0(\xi)d\xi + \int_0^t \int_{\mathbb{R}} \Gamma(x,\xi,t,\tau)f(\xi,\tau)d\xi d\tau$$

is the unique solution of the Cauchy problem (10) *in* $C^{2,1}(\mathbb{R} \times (0,T]) \cap C(\mathbb{R} \times [0,T])$, *with* $|u(x,t)| \leq e^{kx^2}$ *for positive* k.

Continuous dependence on the coefficients:

Definition 2 *Let R, λ and α be positive real numbers, with $0 < \alpha \leq 1$ and $\lambda < R$. We define a vector space for coefficients of L by*
$B(R,\lambda,\alpha) = \{(a(x,t), b(x,t), c(x,t)) \in (C_{\alpha,\frac{\alpha}{2}}(\Omega_T))^3 : \lambda < a, \quad and \quad a, b, c < R\}$ *and for $v, \overline{v} \in B(R,\lambda,\alpha)$, we define*
$\|v - \overline{v}\|_{\alpha,\frac{\alpha}{2}} = max\{\|a - \overline{a}\|^{\Omega_T}_{\alpha,\frac{\alpha}{2}}, \|b - \overline{b}\|^{\Omega_T}_{\alpha,\frac{\alpha}{2}}, \|c - \overline{c}\|^{\Omega_T}_{\alpha,\frac{\alpha}{2}}\}$.

To emphasize the dependence of the L operator on the coefficients $(a, b, c) \equiv v$, we write

$$L_{[v]}u \equiv \frac{\partial u}{\partial t} - a(x,t)\frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial u}{\partial x} + c(x,t)u = 0\,, \qquad (11)$$

with corresponding fundamental solution

$$\Gamma_{[v]}(x,t,\xi,\tau) = Z_{[v]}(x-\xi,\xi,t,\tau) + \int_\tau^t \int_{\mathbb{R}} Z_{[v]}(x-y,y,t,\sigma)\phi_{[v]}(y,\xi,\sigma,\tau)dyd\sigma. \quad (12)$$

Lemma 1 *Given $v, \overline{v} \in B(R,\lambda,\alpha)$, we have that*

$$|(D^s_x Z_{[v]} - D^s_x Z_{[\overline{v}]})(x-\xi,\xi,t,\tau)| \leq K\|a - \overline{a}\|_\infty \frac{1}{(t-\tau)^{\frac{s+1}{2}}} e^{-C\frac{(x-\xi)^2}{(t-\tau)}},$$

for $s = 0, 1, 2$, where $C < \frac{1}{4R}$ is a positive constant and K is also a positive constant depending only on λ.

Lemma 2 *Let $\phi_{[v]}$ and $\phi_{[\overline{v}]}$ be defined in (6), with $v, \overline{v} \in B(R,\lambda,\alpha)$ and $0 \leq \beta \leq 1$. Then, we have the following inequalities,*

$$|(\phi_{[v]} - \phi_{[\overline{v}]})(x,\xi,t,\tau)| \leq K\|v - \overline{v}\|_{\alpha,\frac{\alpha}{2}} \frac{1}{(t-\tau)^{\frac{3-\alpha}{2}}} e^{-C\frac{(x-\xi)^2}{t-\tau}}\,, \qquad (13)$$

and

$$\begin{aligned} &|(\phi_{[v]}(x,\xi,t,\tau) - \phi_{[\overline{v}]}(x,\xi,t,\tau)) - (\phi_{[v]}(y,\xi,t,\tau) - \phi_{[\overline{v}]}(y,\xi,t,\tau))| \\ &\quad \leq K\|v - \overline{v}\|^\beta_{\alpha,\frac{\alpha}{2}} |x-y|^{\alpha(1-\beta)} \frac{1}{(t-\tau)^{\frac{3-\beta\alpha}{2}}} (e^{-C\frac{(x-\xi)^2}{t-\tau}} + e^{-C\frac{(y-\xi)^2}{t-\tau}})\,, \end{aligned} \qquad (14)$$

where $C < \frac{1}{4R}$ and $K = K(R,\lambda,\alpha,T)$ is continuous with respect to T.

Lemma 3 *Let* $v, \overline{v} \in B(R, \lambda, \alpha)$, $0 < \beta < 1$, $\Gamma_{[v]}$ *and* $\Gamma_{[\overline{v}]}$ *fundamental solutions of* $L_{[v]} = 0$ *and* $L_{[v]} = 0$*, respectively. Then,*

$$|(D_x^s\Gamma_{[v]} - D_x^s\Gamma_{[\overline{v}]})(x,t,\xi,\tau)| \le \frac{K\|v-\overline{v}\|_{\alpha,\frac{\alpha}{2}}}{(t-\tau)^{\frac{s+1}{2}}} e^{-C\frac{(x-\xi)^2}{t-\tau}}, \tag{15}$$

$$|(\partial_{xx}\Gamma_{[v]} - \partial_{xx}\Gamma_{[\overline{v}]})(x,t,\xi,\tau)| \tag{16}$$

$\le K(\|v-\overline{v}\|_{\alpha,\frac{\alpha}{2}} + \|v-\overline{v}\|^{\beta}_{\alpha,\frac{\alpha}{2}})(\frac{1}{|x-\xi|^{\frac{2-\alpha}{2}}(t-\tau)^{\frac{2-\alpha}{2}}} + \frac{1}{(t-\tau)^{\frac{3}{2}}})e^{-C\frac{(x-\xi)^2}{t-\tau}}$ *and*

$$|(\partial_t\Gamma_{[v]} - \partial_t\Gamma_{[\overline{v}]})(x,t,\xi,\tau)| \tag{17}$$

$\le K(\|v-\overline{v}\|_{\alpha,\frac{\alpha}{2}} + \|v-\overline{v}\|^{\beta}_{\alpha,\frac{\alpha}{2}})(\frac{1}{|x-\xi|^{\frac{2-\alpha}{2}}(t-\tau)^{\frac{2-\alpha}{2}}} + \frac{1}{(t-\tau)^{\frac{3}{2}}})e^{-C\frac{(x-\xi)^2}{t-\tau}}$,
where $s = 0, 1$, $C \le \frac{1}{4R}$ *and* $K = K(R, \lambda, \alpha, T)$ *is continuous with respects to* T.

Theorem 2 *Let be* $f, \overline{f} \in C_{1,\frac{1}{2}}(\Omega_T)$, $T > 0$ *and* $u_0, \overline{u}_0$ *Lipschitz continuous and bounded real functions. If* u *and* $\overline{u}$ *are, respectively, solutions of*

$$L_{[v]}u = f, \quad \mathbb{R} \times (0,T], \quad u(x,0) = u_0, \quad \mathbb{R}, \tag{18}$$

and

$$L_{[\overline{v}]}u = \overline{f}, \quad \mathbb{R} \times (0,T], \quad u(x,0) = \overline{u}_0 \quad \mathbb{R}, \tag{19}$$

where $v = (a, b, 0), \overline{v} = (\overline{a}, \overline{b}, 0) \in B(R, \lambda, 1)$*, then*

$$\|u-\overline{u}\|_{1,\frac{1}{2}} \le K(\|v-\overline{v}\|_{1,\frac{1}{2}} + \|v-\overline{v}\|^{\beta}_{1,\frac{1}{2}} + \|u_0-\overline{u}_0\|_1 + \tag{20}$$
$$T^{\frac{1}{2}}\max\{\|f\|_{1,\frac{1}{2}}, 1\}(\|f-\overline{f}\|_{1,\frac{1}{2}} + \|v-\overline{v}\|_{1,\frac{1}{2}} + \|v-\overline{v}\|^{\beta}_{1,\frac{1}{2}})),$$

where $K = K(R, \lambda, T, \|u_0\|_1)$ *is continuous with respect to* T.

Corollary 1 *If* u *is a solution of* (18)*, then*
$\|u\|_{1,\frac{1}{2}} \le K(\|v\|_{1,\frac{1}{2}} + \|v\|^{\beta}_{1,\frac{1}{2}} + \|u_0\|_1 + T^{\frac{1}{2}}\max\{\|f\|_{1,\frac{1}{2}}, 1\}(\|f\|_{1,\frac{1}{2}} + \|v\|_{1,\frac{1}{2}} + \|v\|^{\beta}_{1,\frac{1}{2}}))$,
where $K = K(R, \lambda, T, \|u_0\|_1)$.

Lemma 4 *Let consider* $v_n, v \in B(R, \lambda, \alpha)$ *and their respective fundamental solutions,* $\Gamma_{[v_n]}$ *and* $\Gamma_{[v]}$*. If* v_n *converges pointwise to* v*, then* $\Gamma_{[v_n]}$ *converges also pointwise to* $\Gamma_{[v]}$.

Local solution:

Let A be the operator defined by

$$A(u_1, u_2) = (w_1, w_2), \tag{21}$$

where its domain is given in Lemma 5 below, and (w_1, w_2) is the solution of the problem

$$\left\{ \begin{array}{l} L_{[v(u_i)]}(w_i) = F_i(u_1, u_2, y_i), \quad \mathbb{R} \times (0, T] \\ (y_i)_t = -A_i y_i f(u_i), \quad \mathbb{R} \times (0, T] \\ (w_i(x, 0), y_i(x, 0)) = (u_{i,0}(x), y_{i,0}(x)), \quad \mathbb{R}. \end{array} \right. \tag{22}$$

Here, $u_{i,0}$ and $y_{i,0} \geq 0$ are Lipschitz and bounded, $f(u)$ is defined by (3), $v_i(u_i) = (\frac{\lambda_i}{a_i + y_i(u_i)}, \frac{c_i}{a_i + y_i(u_i)}, 0)$ and
$F_i(u_1, u_2, y_i) = \dfrac{b_i A_i u_i + d_i}{a_i + b_i y_i} y_i f(u_i) + (-1)^i q \dfrac{u_1 - u_2}{a_i + b_i y_i}.$

Lemma 5 *Let* $0 < T \leq 1$, $K_i = K(\max\{\frac{\lambda_i}{a_i}, \frac{c_i}{a_i}\}, \frac{\lambda_i}{a_i + b_i \|y_{i,0}\|_\infty}, 1, |u_{i,0}|_1)$ *a constant given by Corollary* (1), $\nu_i = 2(\max\{\frac{\lambda_i}{a_i}, \frac{c_i}{a_i}\} + \max\{\frac{\lambda_i b_i}{a_i^2}, \frac{c_i b_i}{a_i^2}\})$, $M_i > K_i(\nu_i \|y_{i,0}\|_1 + (\nu_i \|y_{i,0}\|_1)^\beta + \|u_{i,0}\|_1)$ *and* $\Sigma = \{(u_1, u_2) \in C_{1,\frac{1}{2}}(\Omega_T) : \|u_i\|_{1,\frac{1}{2}} \leq M_i\}$. *Then, if* T *is sufficiently small the operator* $A : \Sigma \to \Sigma$ *is well defined.*

We proof these results using the integral representation for the solution given in Theorem 1 and the estimates in the last section.

Theorem 3 *If T is sufficiently small the Cauchy problem* (2) *has a solution in* $C^{2,1}(\mathbb{R}\times(0,T])\cap C_{1,\frac{1}{2}}(\Omega_T)$.

Iterative scheme:

$$(w_1^{(n)},w_2^{(n)})=A(w_1^{(n-1)},w_2^{(n-1)})$$

By Arzelà-Ascoli's theorem, there exists a continuous function (u_1,u_2) in $\mathbb{R}\times[0,T]$ and a subsequence of $(w_1^{(n)},w_2^{(n)})$ such that $(w_1^{(n)},w_2^{(n)})$ converges to (u_1,u_2) uniformely in compacts sets.
By the representation formula, we have

$$w_i^{(n+1)}(x,t)= \tag{23}$$

$$\int\Gamma_{[v_i(w_i^{(n)})]}(x,\xi,t,0)u_{i,0}(\xi)d\xi+\int_0^t\int\Gamma_{[v_i(w_i^{(n)})]}(x,\xi,t,\tau)F_i(w_1^{(n)},w_2^{(n)},y_i(w_i^{(n)}))(\xi,\tau)d\xi d\tau,$$

where

$$v_i(w_i^{(n)})=\left(\frac{\lambda_i}{a_i+y_i(w_i^{(n)})},\frac{c_i}{a_i+y_i(w_i^{(n)})},0\right) \tag{24}$$

and

$$y_i(w_i^{(n)})(x,t)=y_{i,0}(x)e^{-A_i\int_0^t f(w_i^{(n)}(x,s))ds}. \tag{25}$$

Global solution

Theorem. *Let* $u_{i,0}$, $i = 1, 2$, *be bounded lipschitzian functions on* $\mathbb{R}$ *and in* $L^p(\mathbb{R})$ *for some* $p \in (1, \infty)$. *Assume also* $y_{i,0} \in C^2 \cap L^\infty$ *and* $y'_{i,0} \in L^\infty$. *Then for any* $T > 0$ *there exists a solution of* (2) *in*

$$C^{2,1}(\mathbb{R} \times (0, T]) \cap C_{1,\frac{1}{2}}(\mathbb{R} \times [0, T]) \cap L^\infty([0, T]; L^p(\mathbb{R})).$$

On the proof:

- We define $[0, T^*)$ as the maximal interval for the local solution $U \equiv (u_1, u_2)$ and show that there exists the limit $\lim_{t \to T^*} U(\cdot, t)$ in the above space.
- To show that U is bounded in $\mathbb{R} \times [0, T^*)$ we use the upper solution mentioned earlier.
- To bound U_x we use the tecnique of

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