On the Complexity of Sandpile Critical Avalanches

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Abstract. In this work we study The Abelian Sandpile Model from the point of view of computational complexity. We begin by studying the length distribution of sandpile avalanches triggered by the addition of two critical configurations: we prove that those avalanches are long on average, their length is bounded below by a constant fraction of the length of the longest critical avalanche which is, in most of the cases, superlinear. At the end of the paper we take the point of view of computational complexity, we analyze the algorithmic hardness of the problem consisting in computing the addition of two critical configurations, we prove that this problem is P complete, and we prove that most algorithmic problems related to The Abelian Sandpile Model are NC reducible to it

Can we quickly predict the evolution of an avalanche if we are given a full description of the initial conditions? *The Abelian Sandpile Model* has been intensively studied in the physics milieu since its introduction by Bak et al [4], this model allows us to simulate dissipative dynamical systems such as forest fires, earth quakes, extinction events, the dynamics of the stock market, and avalanches [3].

One can look at The Abelian Sandpile Model as a special class of graph automata. If one assume this point of view there are several algorithmic issues that one can (has to) take into account. In [8] Moore and Nilsson define *The Sandpile Prediction Problem* which we study in this paper. Moore and Nilsson ask for a characterization of its complexity. In this work we investigate in some depth the algorithmic hardness of The Sandpile Prediction Problem. We show that the prediction problem is reducible to the problem consisting in computing the relaxation of the addition of two critical configurations, and it implies that the later problem, called *The Sandpile Group-computations Problem*, is the hardness core of most algorithmic problems related to The Abelian Sandpile Model.

Previous work and contributions. Moore and Nilsson defined in [8] some computational problems related to The Abelian Sandpile Model, they show that all those problems can be reduced to The Sandpile Prediction Problem. It follows from the work of Tardos [9] that The Sandpile Prediction Problem, (and each one of the problems considered by Moore and Nilsson), is polynomial time solvable when restricted to undirected graphs. Besides of Moore's work there exists some previous work concerning the computational complexity of algorithmic problems related to The Abelian Sandpile Model (see for example [7], [6] and [1]). It is important to remark that our complexity theoretical analysis is based on the notion of NC-Turing reducibility. We have chosen to work with this notion of reducibility because all the algorithmic problems considered in this paper are *Ptime* computable. Furthermore, we are mainly interested in analyzing the complexity of simulating sandpile avalanches, and when we want to use The Abelian Sandpile Model as a model of some dynamical process, we have to consider huge systems, that is: most of the time we have to simulate avalanches occurring on huge graphs, and involving a huge amount of grains of sand. It makes necessary to analyze the *polylogarithmic time simulability* of sandpile avalanches. NC-reducibility notions and NC-completeness notions are the right notions when one has to cope with questions concerning the polylogarithmic time computability of an algorithmic problem.

It is known that one can attach to any sandpile graph (G, S) an abelian group $\mathcal{K}(G)$, which is called the *critical group* of (G, S). The critical group of (G, S) encodes the long term behavior of The Abelian Sandpile Model on (G, S), and its elements are the so called *critical configurations*. It has been argued that The Abelian Sandpile Model is a model of The Self-organized Criticality Theory of Bak et al [4]. If it were the case, critical configurations would be complex configurations, because it would be hard to predict the dynamics triggered by them. There is an algorithmic problem closely related to the computation of the dynamics (*avalanches*) triggered by critical configurations, this problem is GC : *The Sandpile Group-computations Problem.* This work is focused on the complexity analysis of GC. We show that, in the three-dimensional case, the problem GC is the *hardness core* of most algorithmic problems related to The Abelian Sandpile Model. We prove that *The Sandpile Monoid-computations Problem* is NC Turing reducible to GC (then, we have that the prediction problem, the identity problem and the recurrent recognition problem are also NC Turing reducible to GC, see reference [1]). We conjecture that there exist deep links between the hardness of GC and the argued *Self-organized Criticality* of The Abelian Sandpile Model, we would like to make apparent those links and we believe that we have partially fulfilled this goal.

Organization of the work. This work is organized into six sections including the introduction. In section two we introduce The Abelian Sandpile Model and we list some basic facts concerning this model. In sections three and four we study the typical length of the avalanches triggered by the addition of two critical configurations. In section five we introduce some algorithmic problems related to The Abelian Sandpile Model, and we study the relative hardness of those problems. We prove that the Sandpile Prediction Problem is NC reducible to The Sandpile Group-computations Problem. Section six is constituted by some few concluding remarks.

1 The Abelian Sandpile Model

In this section we introduce the basic definitions and some of the basic results concerning The Abelian Sandpile Model.

Definition 1. *A sandpile graph is a pair* (G,S), *where* G *is a finite undirected multigraph and* $S\subseteq V(G)$ *is a nonempty set of nodes satisfying the following condition:*

Given $w\in V(G)-S$, *there exists a path from* w *to some element of* S.

Given (G,S) a sandpile graph, the set S will be called the *sink.* Most of the time we will say that G is a sandpile graph and that S is the sink of G. We will use the symbol $V(G)^*$ to denote the set $V(G)-S$, that is: $V(G)^*$ denotes the set of nodes of G out of the sink. Given $v,w\in V(G)$ we use the symbol E_{vw} to denote the number of edges connecting nodes v and w (recall that G is a multigraph). A *configuration* on G is a function $g:V(G)^*\to\mathbb{N}$. Given g a configuration on G and given $v\in V(G)^*$ we will say that v is *g-stable* if and only if $g(v)\lneqq \deg_G(v)$. We will say that g is an *stable configuration* if and only if for all $v\in V(G)^*$ we have that v is g-stable.

We can attach to any sandpile graph (G,S) a *Graph Automaton* whose underlying graph is G.

Definition 2. *Given* (G,S) *a sandpile graph, the sandpile automaton on* G *is the graph automaton* $SP(G)$ *defined by:*

1. *The set of configurations of* $SP(G)$ *is the set*

$$\{g : g \text{ is a configuration on } G\}$$

2. *Given* g *a configuration of* $SP(G)$ *and given* $v\in V(G)^*$, *the state of* v *under* g *is equal to* $g(v)$.
3. *Given* g *a configuration, the set of possible transitions from* g *is given by the following transition rule:*
   Given $v\in V(G)^*$, *if* $g(v)\geq \deg_G(v)$, *then we have that* $g\to g_v$ *is a possible transition, where* g_v *is the configuration on* G *defined by*

$$g_v(w):=\begin{cases} g(v)-\deg_G(v)\,,\ \text{if } w=v \\ g(w)+E_{vw},\ \text{if } v \text{ is a neighbor of } w \\ g(w)\ \ \text{if } v \text{ is not a neighbor of } w \end{cases}$$

Any transition of $SP(G)$ is called a *firing* or a *toppling.* So, given g a configuration, the transition $g\to g_v$ is a firing, and if such transition occurs we say that node v was fired (toppled) or we say that a firing (toppling) at v has occurred.

Given (G,S) a sandpile graph and given g an initial configuration, we can choose an unstable node, fire it and obtain a new configuration. Note that we can choose any unstable node to produce a firing, in this sense sandpile automata are nondeterministic. A sequence of firings $g_1\to g_2\to ...\to g_n$ is called an *avalanche* of length $n-1$ with initial configuration g_1, and we say that it is an avalanche from g_1 to g_n. If g_n is stable we say that g_n is a *stabilization* or a *relaxation* of g_1.

Remark 1. Given (G,S) a sandpile graph, we use the symbol G to denote it, that is: we will not explicitly mention the sink S.

If we fix a configuration g on G, we can consider the following three sets:

1. $Aval\left(G,g\right)$, the set of avalanches whose initial configuration is g.
2. $Aval_M\left(G,g\right)$, the set of maximal avalanches beginning in g (A is maximal if and only if A can not be extended, that is: A is maximal if and only if its final configuration is stable).
3. $st\left(G,g\right)$ the set of relaxations of g.

Furthermore, given G, g and

$$A = g \to g_1 \to ... \to g_n$$

an avalanche, the *score vector* of A, which we denote with the symbol SC_A, is equal to $\left(t_v\right)_{v\in V(G)^*}$, where for any $v \in V\left(G\right)^*$ the entry t_v is equal to the number of times node v was fired during the occurrence of A.

Theorem 1. *(The fundamental theorem of sandpiles)*
Let G be a sandpile graph and let g be a configuration, we have:

1. *Any avalanche beginning in g is finite.*
2. $\left|st\left(G,g\right)\right| = 1$.
3. *Given A,$B \in Aval_M\left(G,g\right)$, we have that $SC_A = SC_B$.*

A proof of this theorem can be found in [10]. Theorem 1 says many things about sandpile automata. Item 1 says that sandpile automata are *terminating*. Item 2 says that sandpile automata are *confluent*, i.e. the input (the initial configuration) determines an unique output (its stabilization). Item 3 says that, though there are many computation paths, sandpile automata are strongly deterministic: given $SP\left(G\right)$ a sandpile automaton and given two computation paths of $SP\left(G\right)$ on input g, the second path is simply a permutation of the first, and as a consequence they have the same length.

Given $C\left(G\right) = \mathbb{N}^{V(G)^*}$ the set of all the configurations on G and given $st\left(G\right)$ the set of all the stable configurations on G, we can define two functions $st_G : C\left(G\right) \to st\left(G\right)$ and $SC_G : C\left(G\right) \to C\left(G\right)$ in the following way:

1. $st_G\left(g\right) :=$ the stabilization of g.
2. $SC_G\left(g\right) := SC_A$, where A is any element of $Aval_M\left(G,g\right)$.

Note that, for any sandpile graph G, the functions st_G and SC_G are computable, since the avalanches are always finite: given g a configuration on G, if one wants to compute either $st_G\left(g\right)$ or $SC_G\left(g\right)$, one only has to simulate the automaton $SP\left(G\right)$ on input g.

Remark 2. Given $g \in C\left(G\right)$, we use the symbol SC_g to denote the vector $SC_G\left(g\right)$.

We can obtain, as an easy consequence of the invariance of the score vector, the following theorem.

Theorem 2. *Given G a sandpile graph and given f_1, f_2 and f_3 three configurations, we have that*

1. $st_G(f_1 + f_2 + f_3) = st_G(st_G(f_1 + f_2) + f_3)$.
2. $st_G(f_1 + f_2) = st_G(st_G(f_2) + st_G(f_1))$.

Last theorem allow us to associate to any sandpile graph a sandpile monoid. To this end we define a binary operation $\oplus : st(G)^2 \rightarrow st(G)$ in the following way

$$f \oplus g = st_G(f + g)$$

The pair $(st(G), \oplus)$ is a finite commutative monoid. We will use the name *Sandpile Monoid of* G to denote the pair $\mathcal{M}(G) = (st(G), \oplus)$.

Definition 3. *Given $\mathcal{M}$ a finite commutative monoid, the kernel of $\mathcal{M}$ is equal to the intersection of all its nonempty ideals. We use the symbol $Ker(\mathcal{M})$ to denote the kernel of $\mathcal{M}$.*

Remark 3. Observe that, if $\mathcal{M}$ is a finite commutative monoid, $Ker(\mathcal{M})$ is a nonempty ideal of $\mathcal{M}$.

It is known that the *kernel* of a finite commutative monoid is an abelian group (see reference [10], [1]). We use the symbol $\mathcal{K}(G)$ to denote the abelian group

$$\left(Ker(\mathcal{M}(G)), \oplus \restriction_{(Ker(\mathcal{M}(G)))^2}\right)$$

which we call *the critical group* (or the sandpile group) of G. The elements of $\mathcal{K}(G)$ are the so called *critical* (*recurrent*) configurations. The binary operation of $\mathcal{K}(G)$ is defined by the following equation.

$$st_G(f + g) = st_G(f) \oplus st_G(g).$$

2 The length of critical avalanches: bounded classes

We will use the term *critical avalanches* to denote the avalanches triggered by the addition of two critical configurations. In this section we study the length of critical avalanches, that is: we establish upper and lower bounds on the possible length of critical avalanches.

Definition 4. *A bounded class of sandpile graphs is a class $\mathcal{C}$ of sandpile graphs for which there exists $D_{\mathcal{C}} \geq 1$ such that for any $G \in \mathcal{C}$ and for all $v \in V(G)^*$ we have that* $\deg_G(v) \leq D_{\mathcal{C}}$.

Given f, g two configurations on G, we use the symbol $f \leq g$ to indicate that for all $v \in V(G)^*$, we have that $f(v) \leq g(v)$.

Given $f, g \in \mathcal{K}(G)$ we will use the symbol $L(f, g)$ to denote the length of the critical avalanches triggered by $f + g$. Note that

1. If $f \leq g$ and $h \leq r$, then $L(f,h) \leq L(g,r)$.
2. Let M_G be the *maximal configuration* on G, which is the configuration defined by: $M_G(v) = \deg_G(v) - 1$. We have that M_G is critical. Furthermore, we have that for any $f, g \in \mathcal{K}(G)$ the inequality $L(f,g) \leq L(M_G, M_G)$ holds.

Given G a sandpile graph, we use the symbol $\beta(G)$ to denote the set

$$\left\{w \in V(G)^* : \exists w \in S(\{v,w\} \in E(G))\right\}$$

We use the symbol β_G to denote the configuration defined by: given $v \in V(G)^*$ we have that

$$\beta_G(v) = \sum_{s \in S} E_{vs}$$

We will use many times the following theorem, which is indebted to D. Dhar (A proof of this theorem can be found in [5]).

Theorem 3. *(Dhar's theorem)*
Let G be an undirected sandpile graph and let $f \in C(G)$.

1. *f belongs to $\mathcal{K}(G)$ if and only if $SC_{f+\beta_G}(v) = 1$ for any $v \in V(G)^*$.*
2. *f belongs to $\mathcal{K}(G)$ if and only if $f \oplus \beta_G = f$.*
3. *If $f \in \mathcal{K}(G)$, then for any $\{v,w\} \in E(G)$ we have that either $f(v) \gneqq 0$ or $f(w) \gneqq 0$.*

Let $\mathcal{C}$ be a bounded class. We prove that the critical avalanches occurring on $\mathcal{C}$-graphs can not be short, their length can not be sublinear.

Theorem 4. *(Critical configurations can only generate long avalanches)*
Given $G \in \mathcal{C}$ and given $f \in \mathcal{K}(G)$ we have

$$\forall g \in \mathcal{K}(G) \left(L(f,g) \geq \frac{\left|V(G)^*\right| - D_{\mathcal{C}} \left|\beta(G)\right|}{D_{\mathcal{C}}} \right)$$

Proof. Let $H(G) = \sum_{v \in V(G)^*} (\deg_G(v) - 1)$. Remember that a configuration f is a recurrent configuration if and only if

1. for any $v \in V(G)^*$ we have $SC_{f+\beta_G}(v) = 1$.
2. $st_G(f + \beta_G) = f$.

Suppose that we run the avalanche triggered by $f + \beta_G$ and for any $v \in V(G)^*$ we count the number of grains on v just before the node v is toppled. Let α be equal to the result of our counting. Note that $\alpha \geq H(G) + \left|V(G)^*\right|$. On the other hand, it is easy to verify that we count twice the grains which remain on $V(G)^*$ after the avalanche, and we count once the lost grains. So we have

$$2\|f\| + D_{\mathcal{C}} \left|\beta(G)\right| \geq \alpha \geq H(G) + \left|V(G)^*\right|$$

Thus, we have that

$$\|f\| \geq \left(\frac{H(G) + \left(\left| V(G)^* \right| - D_{\mathcal{C}} \left| \beta(G) \right| \right)}{2} \right)$$

Now, given $f, g \in \mathcal{K}(G)$ we have that

$$\|f\| + \|g\| \geq H(G) + \left(\left| V(G)^* \right| - D_{\mathcal{C}} \left| \beta(G) \right| \right)$$

and it implies that, when we begin with the configuration $f + g$, we have to throw at least $\left(\left| V(G)^* \right| - D_{\mathcal{C}} \left| \beta(G) \right| \right)$ grains. We can throw at most $D_{\mathcal{C}}$ grains per toppling, and it implies that

$$L(f, g) \geq \frac{\left| V(G)^* \right| - D_{\mathcal{C}} \left| \beta(G) \right|}{D_{\mathcal{C}}}$$

Corollary 1. *Suppose that for any $G \in \mathcal{C}$ we have that $K \geq \left| \delta(G) \right|$, then for any $f, g \in \mathcal{K}(G)$*

$$L(f, g) \geq \frac{\left| V(G)^* \right| - D_{\mathcal{C}} K}{D_{\mathcal{C}}} \in \Omega\left(\left| V(G)^* \right| \right)$$

Now, we will establish a lower bound on $L(w_G, w_G)$ which could be stronger than the linear bound of theorem 4. Let G be an element of $\mathcal{C}$ whose sink is equal to S, the symbol $\rho(G)$ denotes the quantity $\max_{v \in V(G)^*} \{d_G(v, S)\}$, where $d_G(v, S)$ denotes the distance from v to the sink of G. Recall that the distance from v to S is equal to the number of edges that constitutes the shortest path connecting v and S.

Theorem 5. $L(M_G, M_G) \in \Omega\left(\left| V(G)^* \right| + \rho(G)^2 \right)$.

Proof. Let $\mathcal{C}$ be a bounded class and let G be an element of $\mathcal{C}$. Recall that all the avalanches triggered by $2M_G$ have the same length. We want to lowerbound the length of a very specific avalanche triggered by $2M_G$. Given $i \leq \rho(G)$, we use the symbol $N_i(G)$ to denote the induced subgraph of G whose vertex set is equal to

$$\{v \in V(G) : d_G(v, S) \geq i\}$$

We note that for all $1 \leq i \leq \rho(G)$ the graph $N_i(G)$ is embedded in $N_{i-1}(G)$, and we note that $N_0(G)$ is equal to G. Given $i \geq 1$, we can think of $N_i(G)$ as a sandpile graph whose sink is equal to the border of $N_{i-1}(G)$. We use the symbol M_i to denote the maximal configuration on $N_i(G)$, and we use the symbol β_i to denote the *border configuration* $\beta_{N_i(G)}$. We can express the configuration M_{i-1} as $M_i + \beta_i + \gamma_i$, where γ_i is some configuration on $N_{i-1}(G)$. Note that

$$2M_{i-1} = (M_i + \beta_i) + (M_{i-1} + \gamma_i)$$

We know that that

$$st_{N_{i-1}(G)}\left(2M_{i-1}\right)=st_{N_{i-1}(G)}\left(st_{N_{i-1}(G)}\left(M_i+\beta_i\right)+st_{N_{i-1}(G)}\left(M_{i-1}+\gamma_i\right)\right)$$
$$st_{N_i(G)}\left(M_i+\beta_i\right)=M_i \text{ and } L\left(M_i,\beta_i\right)=\left|N_i\left(G\right)\right|$$

Thus, we have that there exists a configuration γ_2 such that we can pass from the configuration $2M_1=2M_G$ to the configuration $2M_2+\gamma_2$. Furthermore, we have that the partial avalanche carrying us from $2M_1$ to $2M_2+\gamma_2$ has a length which is bounded below by $\left|N_1\left(G\right)\right|$. This partial avalanche (it is not a maximal avalanche) is the first stage of the whole stabilization process. In the second stage we work on the subgraph $N_2\left(G\right)$ with the configuration $2M_2$. We can claim that after $\left|N_2\left(G\right)\right|$ topplings we can pass from $2M_2$ to $2M_3+\gamma_3$, where γ_2 is some configuration on $N_2\left(G\right)$. If we continue in this way, going to the *core* (*center*) of G, we have to generate $\rho\left(G\right)-1$ partial avalanches whose lengths are lowerbounded by $\left|N_1\left(G\right)\right|,\left|N_2\left(G\right)\right|,...,\left|N_{\rho(G)-1}\left(G\right)\right|$ (respectively). Therefore, we have that $L\left(M_G,M_G\right)\geq\left(\sum_{i=1}^{\rho(G)-1}\left|N_i\left(G\right)\right|\right)$. To finish with the proof we observe that:

1. $\left|N_1\left(G\right)\right|=\left|V\left(G\right)^*\right|$.
2. For all $i\lneqq\rho\left(G\right)$ we have that $\left|N_{i+1}\left(G\right)\right|\lneqq\left|N_i\left(G\right)\right|$.

Thus we have that $L\left(M_G,M_G\right)\in\Omega\left(\left|V\left(G\right)^*\right|+\rho\left(G\right)^2\right)$

Corollary 2. *If $\mathcal{C}$ is a bounded class of sandpile graphs such that $\rho\left(G\right)\notin O\left(\sqrt{\left|V\left(G\right)\right|^*}\right)$, then $L\left(w_G,w_G\right)\notin O\left(\left|V\left(G\right)\right|^*\right)$.*

2.1 Sandpile lattices: an example

Given $n,m\geq 1$ we use the symbol $\mathcal{G}_n^m$ to denote the n-dimensional lattice of order m, which is the graph defined by:

- $V\left(\mathcal{G}_n^m\right)=\left[m\right]^n$, where $[m]$ denotes the set $\{1,...,m\}$.
- $E\left(\mathcal{G}_n^m\right)$ is the set constituted by the pairs

$$\left\{\left\{\left(x_1,...,x_n\right),\left(y_1,...,y_n\right)\right\}\in\left(\left[m\right]^n\right)^2:\sum_{i=1}^{n}\left|x_i-y_i\right|=1\right\}$$

Given $n,m\geq 1$ the m-dimensional sandpile lattice of order n is the sandpile graph defined by:

- $V\left(\mathcal{L}_n^m\right)=V\left(\mathcal{G}_n^m\right)\cup\{s\}$, where s (the sink of $\mathcal{L}_n^m$) doesn't belong to $V\left(\mathcal{G}_n^m\right)$.
- Any edge of $\mathcal{G}_n^m$ is also an edge of $\mathcal{L}_n^m$. Furthermore, given $v\in V\left(\mathcal{G}_n^m\right)$ we add $2n-\deg_{\mathcal{G}_n^m}\left(v\right)$ edges connecting node v with s.

Note that for all $m, n \geq 1$ and for any $v \in V\left(\mathcal{G}_n^m\right)$ the equation $\deg_{\mathcal{L}_n^m}(v) = 2n$ holds.

Given $n \geq 1$, we use the symbol $\mathcal{L}_n$ to denote the sandpile class $\{\mathcal{L}_n^m : m \geq 1\}$. We note that $\mathcal{L}_n$ is a bounded class of sandpile graphs. Now we will establish an upper bound on the length of the critical avalanches that can occur on $\mathcal{L}_n$. Suppose that we have fixed a natural number $n \geq 2$.

Theorem 6. *Given $n \geq 2$ we have that $L\left(M_{\mathcal{L}_n^m}, M_{\mathcal{L}_n^m}\right) \in \Omega\left(\left|\mathcal{L}_n^m\right|^{\frac{n+1}{n}}\right)$.*

Proof. We have, from the proof of theorem 5, that

$$L\left(M_{\mathcal{L}_n^m}, M_{\mathcal{L}_n^m}\right) \geq \sum_{i=1}^{d(\mathcal{L}_n^m)} \left|N_i\left(\mathcal{L}_n^m\right)\right|$$

We note that

1. $d\left(\mathcal{L}_n^m\right) \geq \left\lfloor \frac{m}{2} \right\rfloor$.
2. $\left|N_i\left(\mathcal{L}_n^m\right)\right| \geq (m-2i)^n$

Thus, we have that

$$L\left(M_{\mathcal{L}_n^m}, M_{\mathcal{L}_n^m}\right) \geq \sum_{i=1}^{d(\mathcal{L}_n^m)} \left|N_i\left(\mathcal{L}_n^m\right)\right| \geq \sum_{i=1}^{\left\lfloor \frac{m}{2} \right\rfloor} (m-2i)^n \in \Omega\left(m^{n+1}\right) = \Omega\left(\left|\mathcal{L}_n^m\right|^{\frac{n+1}{n}}\right)$$

Remark 4. Moore and Nilsson proved that $L\left(M_{\mathcal{L}_n^m}, M_{\mathcal{L}_n^m}\right) \in O\left(\left|\mathcal{L}_n^m\right|^{\frac{n+2}{n}}\right)$.

3 Self organized criticality

The Abelian Sandpile Model has attracted the attention of many researchers, working in the fields of statistical mechanics and complex systems, because it is the toy model of The Self-organized Criticality Paradigm introduced by Bak et al in [4]. According to Bak, a dynamical system exhibits self-organized criticality if it always evolves towards critical states without fine tuning on some control parameters. We will say that a system has the *SOC* property if and only if the system is a model of The Self-organized Criticality Theory of Bak. Thus, we have that critical states are *dense* for systems for which the *SOC* property holds. Let G be a sandpile graph and let $\mathcal{C}(G)$ be the graph of stable configurations defined by:

- $V(\mathcal{C}(G)) = st(G)$.
- Given $f, g \in st(G)$ we have that $(f, g) \in E(\mathcal{C}(G))$ if and only if there exists $v \in V(G)^*$ such that $f \oplus e_v = g$.

Consider the Markov Chain $(\mathcal{C}(G), \mathcal{T})$, where $\mathcal{T}$ is the transition mechanism defined by:

1. We choose uniformly at random $v \in V(G)$.
2. Given $X_n = f$ we set $X_{n+1} = f \oplus e_v$.

The set of *Markov-recurrent configurations*, that is the set of configurations which, with probability 1, are visited infinite many times, is equal to the set of recurrent (critical) configurations (see references [9], [2] and [1]). Also, the set of critical configurations of a sandpile graph G can be considered as the long term behavior of The Abelian Sandpile Model on G. Furthermore, the Markov Chain $(st(G), \mathcal{T})$ has to enter $\mathcal{K}(G)$ after a number of iterations which doesn't depend on the initial configuration, and once the Markov Chain enters $\mathcal{K}(G)$ it can not leave $\mathcal{K}(G)$, that is: the dynamics of The Abelian Sandpile Model spontaneously evolves towards critical states (configurations). But, are critical configurations really critical? The notion of critical state has been associated with the emergence of power law distributions [3]. In [5] Dhar implicitly introduced the following notion of criticality:

Definition 5. *An infinite sandpile graph $\mathcal{G}$ is critical if and only there exist C and α such that for all $n \geq 0$ we have that*

$$\Pr_{f,g \in \mathcal{K}(G)} \left[L(f,g) \geq n \right] \sim C n^{-\alpha}$$

We will identify the notion of criticality with a qualitative property which have been observed of the avalanches triggered by the sum of two critical configurations, and which is implied by the emergence of power law distributions.

Definition 6 (Critical configurations generate, very often, long avalanches). *We say that a class $\mathcal{C}$ of sandpile graphs is a critical class if and only if there exist $\alpha, K \gneqq 0$ such that for any $G \in \mathcal{C}$ we have*

$$\Pr_{f,g \in \mathcal{K}(G)} \left[L(f,g) \geq \frac{L(w_G, w_G)}{\alpha} \right] \geq K$$

We will prove that if we choose uniformly at random two critical configurations f and g, then with high probability the avalanche triggered by $f+g$ is large, its length is almost equal to the length of the longest critical avalanche. First at all we have to introduce a notion of accessibility between configurations. Given G a sandpile graph, we use the symbol $L(G)$ to denote the reduced laplacian of G (see references [1] and [10]). Given $f, g \in C(G)$ we say that g is *accessible* from f if and only if there exists a configuration $h \geq g$ and there exists a configuration t such that

$$h = f + (L(G))(t)$$

We will use the symbol $f \rightarrow g$ to indicate that g is accessible from f. Note that g is accessible from f if and only if there exists a configuration $h \geq g$ such that if we begin with f, we can choose a sequence of nodes, topple those nodes according to the order established by the sequence, and obtain h.

Lemma 1. *Let $\mathcal{C}$ be a bounded class of sandpile graphs and let $G \in \mathcal{C}$, we have that for any $f_1, ..., f_{2(D_\mathcal{C})^2} \in \mathcal{K}(G)$ the configuration $2M_G$ is accessible from $f_1 + ... + f_{2(D_\mathcal{C})^2}$*

Proof. Remember that given $f \in \mathcal{K}(G)$ and given $\{v, w\} \in E(G)$, either $f(w) \gneqq 0$ or $f(v) \gneqq 0$, (see reference [5]). Let $f_1, ..., f_{D_\mathcal{C}+1}$ be $D_\mathcal{C}+1$ critical configurations, given $v \in V(G)^*$ we have that either there exists $i \leq D_\mathcal{C}+1$ such that $f_i(v) \gneqq 0$ or for any w neighbor of v and for any $i \leq D_\mathcal{C}+1$ we have that $f_i(w) \gneqq 0$. Suppose that for all $i \leq D_\mathcal{C}+1$ we have that $f_i(v) = 0$, in this case we can choose any neighbor of v, say w, and fire it. Also, we can place at least one chip on v, taking care of leaving at least one chip on w. It is clear that if we begin with the configuration $\sum\limits_{i \leq D_\mathcal{C}+1} f_i$ we can choose a sequence of at most $\left|V(G)^*\right|$ topplings to obtain a configuration h such that for any $v \in V(G)^*$ the configuration h takes a nonzero value on v. Then, given $f_1, ..., f_{2(D_\mathcal{C})^2} \in \mathcal{K}(G)$ we have that $\sum\limits_{i \leq 2(D_\mathcal{C})^2} f_i \rightarrow 2M_G$.

Theorem 7. *(critical configurations generate, with high probability, long avalanches) Let $\mathcal{C}$ be a bounded class of sandpile graphs and let $G \in \mathcal{C}$, we have that*

$$\Pr_{f,g \in \mathcal{K}(G)}\left[L(f,g) \geq \frac{L(M_G, M_G)}{4^{(D_\mathcal{C})^2}}\right] \geq \frac{1}{2(D_\mathcal{C})^2}$$

Proof. Given $f_1, f_2, ..., f_{2(D_\mathcal{C})^2}$ we have that $\sum\limits_{i \leq 2(D_\mathcal{C})^2} f_i \rightarrow 2M_G$. It implies that

$$L\left(f_{2(D_\mathcal{C})^2}, \sum_{i \leq 2(D_\mathcal{C})^2 - 1} f_i\right) \geq L(M_G, M_G)$$

Also, we have that either $\left(L\left(f_{2(D_\mathcal{C})^2}, \bigoplus\limits_{i \leq 2(D_\mathcal{C})^2-1} f_i\right) \geq \frac{L(M_G, M_G)}{2}\right)$ or $\left(L\left(f_{2(D_\mathcal{C})^2-1}, \sum\limits_{i \leq 2(D_\mathcal{C})^2-2} f_i\right) \geq \frac{L(M_G, M_G)}{2}\right)$.

Arguing in this way we can prove that there exists $i \leq 2(D_\mathcal{C})^2$ such that

$$L\left(f_i, \bigoplus_{j \leq i-1} f_j\right) \geq \frac{L(M_G, M_G)}{4^{(D_\mathcal{C})^2}}$$

Thus, we have that

$$\Pr_{f_1, ..., f_{2(D_\mathcal{C})^2}}\left[\exists i \leq 2(D_\mathcal{C})^2 \left(L\left(f_i, \bigoplus_{j \leq i-1} f_j\right) \geq \frac{L(M_G, M_G)}{4^{(D_\mathcal{C})^2}}\right)\right] = 1$$

Note that for any $f \in \mathcal{K}(G)$ and for any $i \geq 1$ we have that

$$\Pr_{f_1,...,f_i}\left[\bigoplus_{j\leq i} f_j = f\right] = \frac{1}{|\mathcal{K}(G)|}$$

Given $f_1, ..., f_\alpha$ a sequence of critical configurations on G and given $i \leq \alpha - 1$, we define $g_i = \bigoplus_{j\leq i} f_j$. We have that:

1. The procedure below is a sound method to generate, uniformly at random, two elements of $\mathcal{K}(G)$.
   - Choose uniformly at random $f_1, ..., f_\alpha$, $(\alpha \geq 2)$.
   - Choose uniformly at random $i \in \{2, ..., \alpha\}$.
   - Compute f_i and g_{i-1}.
2. It holds that

$$\Pr_{f_1,...,f_{2(D_\mathcal{C})^2}}\left[\exists 2 \leq i \leq 2(D_\mathcal{C})^2 \left(L(f_i, g_{i-1}) \geq \frac{L(M_G, M_G)}{4^{(D_\mathcal{C})^2}}\right)\right] = 1$$

From items 1 and 2 we obtain

$$\Pr_{f,g\in\mathcal{K}(G)}\left[L(f,g) \geq \frac{L(M_G, M_G)}{4^{(D_\mathcal{C})^2}}\right] =$$

$$\Pr_{2\leq i\leq 2(D_\mathcal{C})^2;\ f_1,...,f_{2(D_\mathcal{C})^2}}\left[L(f_i, g_{i-1}) \geq \frac{L(M_G, M_G)}{4^{(D_\mathcal{C})^2}}\right] \geq \frac{1}{2(D_\mathcal{C})^2}$$

Thus, we haven proven that

$$\Pr_{f,g\in\mathcal{K}(G)}\left[L(f,g) \geq \frac{L(M_G, M_G)}{4^{(D_\mathcal{C})^2}}\right] \geq \frac{1}{2(D_\mathcal{C})^2}$$

Summarizing we have

Theorem 8. *If $\mathcal{C}$ is a bounded class of sandpile graphs, then $\mathcal{C}$ is a critical class.*

3.1 Sandpile lattices are superlinear

If we restrict our attention to the case of sandpile lattices (the most prominent examples of bounded classes) we can obtain some more specific results.

Theorem 9. *Let $n \geq 2$, we have*

1. *Given $m \geq 1$ we have*

$$\Pr_{f,g\in\mathcal{K}(\mathcal{L}_n^m)}\left[L(f,g) \geq \frac{L\left(M_{\mathcal{L}_n^m}, M_{\mathcal{L}_n^m}\right)}{4^{(2d)^2}}\right] \geq \frac{1}{2(2d+1)(2d-1)}$$

2. The function $\Psi_n : \mathbb{N} \to \mathbb{N}$ defined by

$$\Psi_n(m) = \frac{L\left(M_{\mathcal{L}_n^m}, M_{\mathcal{L}_n^m}\right)}{4^{(2d)^2}}$$

is superlinear.

Given $\mathcal{C}$ a bounded class, we say that $\mathcal{C}$ is *superlinear* if and only if there exists a superlinear function $\Psi_{\mathcal{C}}$ such that the probability of choosing $G \in \mathcal{C}$ and two critical configurations $f, g \in \mathcal{K}(G)$ for which $L(f, g) \geq \Psi_{\mathcal{C}}(|G|)$ is bounded below by a positive constant. Given $n \geq 2$, we have proven that $\mathcal{L}_n$ is a superlinear class.

4 The Hardness of sandpile group computations

In this section we analyze the algorithmic complexity of some problems related to The Abelian Sandpile Model. First at all we define *The Sandpile Prediction Problem*, we use the symbol SPP to denote it. Given $\mathcal{C}$ a class of sandpile graphs, we define the prediction problem $SPP[\mathcal{C}]$, which is the restriction of SPP to $\mathcal{C}$-graphs, in the following way

Problem 1. ($SPP[\mathcal{C}]$, sandpile prediction for C-graphs)

- *Input: A sandpile graph $G \in C$ and an initial configuration $g \in C(G)$.*
- *Problem: Compute $st_G(g)$.*

Tardos' bound [9] implies that SPP and each one of the algorithmic problems introduced below can be solved in polynomial time, because of this we analyze the relative complexity of those problems using the notion of NC reducibility.

Definition 7. *Given L and L^* two algorithmic problems, the problem L is NC Turing reducible to the problem L^* if and only if there exists an algorithm $\mathcal{N}$ such that:*

1. *$\mathcal{N}$ has oracle access to L^*.*
2. *There exists $i \geq 1$ such that $\mathcal{N}$ solves the problem L in time $O\left(\log^i\right)$, using a polynomial number of processors and querying the L^*-oracle at most $O\left(\log^i\right)$ times.*

Remark 5. Let L and L^* be two algorithmic problems, if L is NC reducible to L^* and L is P complete, then L^* is P complete, that is: if L is NC reducible to L^* and L can not be solved in polylogarithmic time, then L^* can not be solved in polylogarithmic time.

Now, we will introduce two algebraic problems related to The Abelian Sandpile Model. Let $\mathcal{C}$ be a class of sandpile graphs. First at all we introduce *The Monoid-computations Problem.*

Problem 2. ($MC\left[\mathcal{C}\right]$, mixed computations for C-graphs)

- *Input: (G, f, g), where $G \in C$ and $f \in K(G), g \in M(G)$.*
- *Problem: Compute $f \oplus g$.*

We will focus our research on the algorithmic problem GC, which is the restriction of MC to critical (also called recurrent) configurations. Let $\mathcal{C}$ be a class of sandpile graphs.

Problem 3. ($GC\left[\mathcal{C}\right]$, group computations for C-graphs)

- *Input: (G, f, g), where $G \in C$ and $f, g \in K(G)$.*
- *Problem: Compute $f \oplus g$.*

We begin by analyzing the algorithmic hardness of $GC\left[\mathcal{L}_n\right]$, we prove that if $n \geq 3$ the problem $GC\left[\mathcal{L}_n\right]$ is P complete. First, we have to introduce an additional problem.

The n-dimensional Recurrence Recognition Problem, is the algorithmic problem defined by:

Problem 4. ($RR\left[\mathcal{L}_n\right]$*; recurrence recognition*)

- *Input: (m, g), where $m \in \mathbb{N}$ and $g \in \mathcal{M}(\mathcal{L}_n^m)$.*
- *Problem: decide if $g \in \mathcal{K}(\mathcal{L}_n^m)$.*

Remark 6. Tardos' bound [9] and Dhar's burning algorithm [5] imply that $RR\left[\mathcal{L}_n\right]$, and each one of the algorithmic problems introduced in this section, can be solved in polynomial time.

We will prove that $RR\left[\mathcal{L}_n\right]$ is NC Turing reducible to $GC\left[\mathcal{L}_n\right]$. Given $\mathcal{L}_n^m$ a n-dimensional sandpile lattice, we use the symbol $e_{\mathcal{L}_n^m}$ to denote the identity of $\mathcal{K}(\mathcal{L}_n^m)$. Consider the following two problems

Problem 5. ($Id\left[\mathcal{L}_n\right]$, computation of n-dimensional identities)

- *Input: m, where m is a positive integer.*
- *Problem: compute $e_{\mathcal{L}_n^m}$.*

Problem 6. ($In\left[\mathcal{L}_n\right]$, computation of n-dimensional inverses)

- *Input: (m, g), where m is a positive integer and $g \in K(\mathcal{L}_n^m)$.*
- *Problem: compute g^{-1}.*

Lemma 2. *Let $n \geq 2$, we have*

1. *$Id\left[\mathcal{L}_n\right]$ is NC reducible to $GC\left[\mathcal{L}_n\right]$.*
2. *$In\left[\mathcal{L}_n\right]$ is NC reducible to $GC\left[\mathcal{L}_n\right]$.*

Proof. Let $n, m \geq 1$, we can compute $e_{\mathcal{L}_n^m}$ in constant time using an oracle for $GC\left[\mathcal{L}_n\right]$, to this end we can use the following equations:

1. $\left(M_{\mathcal{L}_n^m}\right)^{-1} = M_{\mathcal{L}_n^m} - \left(M_{\mathcal{L}_n^m} \oplus M_{\mathcal{L}_n^m}\right).$
2. $e_{\mathcal{L}_n^m} = M_{\mathcal{L}_n^m} \oplus \left(M_{\mathcal{L}_n^m}\right)^{-1}.$

Thus, we have that $Id\left[\mathcal{L}_n\right]$ is NC reducible to $GC\left[\mathcal{L}_n\right].$

To finish with the proof, we check that if oracle access to $GC\left[\mathcal{L}_n\right]$ is provided, then $In\left[\mathcal{L}_n\right]$ can be computed in time $O\left(\log\left(m\right)\right),$ using a polynomial number of processors and querying the oracle at most $O\left(\log\left(m\right)\right)$ times.

Given $v \in V\left(\mathcal{L}_n^m\right)^*$, we use the symbol e_v to denote the configuration

$$e_v\left(w\right) = \begin{cases} 1 \text{ if } v = w \\ 0, \text{ otherwise} \end{cases}$$

Let $v \in V\left(\mathcal{L}_n^m\right)^*$ and let $f_v = M_{\mathcal{L}_n^m} - e_v$. It follows from theorem 3 that f_v is a critical configuration. We observe that

$$g^{-1} = \left(\bigoplus_{v \in V\left(\mathcal{L}_n^m\right)^*} g\left(v\right) f_v\right) \oplus \left(\underbrace{\left(M_{\mathcal{L}_n^m}\right)^{-1} \oplus \ldots \oplus \left(M_{\mathcal{L}_n^m}\right)^{-1}}_{\|g\| \text{ times}}\right)$$

It should be clear that we can compute the term on the right hand side of the above equation in time $O\left(\log\left(m\right)\right)$, using a polynomial number of processors, and querying the $GC\left[\mathcal{L}_n\right]$ oracle at most $O\left(\log\left(m\right)\right)$ times. Thus, we have proven that the computation of n-dimensional inverses is NC reducible to $GC\left[\mathcal{L}_n\right].$

We introduce an additional problem, which will be used in the definition of our NC reduction of $RR\left[\mathcal{L}_n\right]$ in $GC\left[\mathcal{L}_n\right].$ Let $\epsilon^{(n,m)} : V\left(\mathcal{L}_n^m\right)^* \rightarrow \mathcal{K}\left(\mathcal{L}_n^m\right)$ be the function defined by $\epsilon^{(n,m)}\left(v\right) = e_{\mathcal{L}_n^m} \oplus e_v$ and let $\epsilon_n : \mathbb{N}^n \times \mathbb{N} \rightarrow \left(\bigcup_{i \geq 1} \mathcal{K}\left(\mathcal{L}_n^m\right)\right) \cup \{\infty\}$ be the function defined by

$$\epsilon_n\left(v, m\right) = \begin{cases} \epsilon^{(n,m)}\left(v\right) \text{ if } v \in V\left(\mathcal{L}_n^m\right)^* \\ \infty, \text{ else} \end{cases}$$

Problem 7. *($EC\left[\mathcal{L}_n\right]$; computation of ϵ_n)*

- *Input:* $\left(m, v\right),$ *where* $m \in \mathbb{N}$ *and* $v \in V\left(\mathcal{L}_n^m\right)^*.$
- *Problem: Compute* $\epsilon_n\left(v, m\right)$.

Next lemma is the main technical result of this section.

Lemma 3. *Let* $n \geq 2$, *we have*

1. *$MC\left[\mathcal{L}_n\right]$ can be computed in time $O\left(\log^2\left(m\right)\right)$ if oracle access to $EC\left[\mathcal{L}_n\right]$ and $GC\left[\mathcal{L}_n\right]$ is provided.*
2. *$EC\left[\mathcal{L}_n\right]$ is NC Turing reducible to $GC\left[\mathcal{L}_n\right].$*
3. *$RR\left[\mathcal{L}_n\right]$ is NC reducible to $GC\left[\mathcal{L}_n\right]$.*

Proof. 1. (proof of item 1) Let $(\mathcal{L}_n^m, f, g)$ be an instance of $MC[\mathcal{L}_n]$. We observe that

$$f \oplus g = f \oplus g \oplus \underbrace{e_{\mathcal{L}_n^m} \oplus ... \oplus e_{\mathcal{L}_n^m}}_{\|g\|\text{-times}}$$

If we express g as $\sum_{v \in V(\mathcal{L}_n^m)^*} g(v)\, e_v$ we get

$$f \oplus g = f \oplus \left(\bigoplus_{v \in V(\mathcal{L}_n^m)^*} g(v)\, \epsilon^{(n,m)}(v) \right)$$

Also, we can use m^3 processors to compute $\left\{ g(v)\, \epsilon^{(n,m)}(v) \right\}_{v \in V(\mathcal{L}_n^m)^*}$, this computation takes $O\left(\log^2(m + \|g\|)\right)$ time units, since we are supposing that we have oracle access to $EC[\mathcal{L}_n]$. We can use the same m^3 processors to compute $f \oplus \left(\bigoplus_{v \in V(\mathcal{L}_n^m)^*} g(v)\, \epsilon^{(n,m)}(v) \right)$ in time $O\left(\log^2(m + \|f\| + \|g\|)\right)$, since we are supposing the we have oracle access to $GC[\mathcal{L}_n]$.

2. (proof of item 5) Let $v \in V(\mathcal{L}_n^m)^*$ and let $f_v = M_{\mathcal{L}_n^m} - e_v$. It follows from theorem 3 that f_v is a critical configuration. Now, we observe that

$$\epsilon^{(n,m)}(v) = e_v \oplus e_{\mathcal{L}_n^m} = e_v \oplus \left(f_v \oplus (f_v)^{-1} \right) = (e_v \oplus f_v) \oplus (f_v)^{-1} = M_{\mathcal{L}_n^m} \oplus (f_v)^{-1}$$

Thus, if one wants to compute $\epsilon^{(n,m)}(v)$, one only has to compute $M_{\mathcal{L}_n^m} \oplus (f_v)^{-1}$ (note that $M_{\mathcal{L}_n^m}, (f_v)^{-1} \in \mathcal{K}(\mathcal{L}_n^m)$). We can compute $(f_v)^{-1}$ in time $O(\log(m))$. Thus, we can solve the problem $EC[\mathcal{L}_n]$ in time $O(\log(m))$ if oracle access to $GC[\mathcal{L}_n]$ is provided.

3. We recall that $\mathcal{K}(\mathcal{L}_n^m)$ is an ideal of $\mathcal{M}(\mathcal{L}_n^m)$, it implies that for all $g \in \mathcal{M}(\mathcal{L}_n^m)$, configuration g belongs to $\mathcal{K}(\mathcal{L}_n^m)$ if and only if $g \oplus e_{\mathcal{L}_n^m} = g$. Also, in order to determine if g belongs to $\mathcal{K}(\mathcal{L}_n^m)$, it is sufficient to compute $g \oplus e_{\mathcal{L}_n^m}$, which can be accomplished in constant time by making only three queries to the $MC[\mathcal{L}_n]$ oracle (compute $M_{\mathcal{L}_n^m} - \left(M_{\mathcal{L}_n^m} \oplus M_{\mathcal{L}_n^m}\right)$, $M_{\mathcal{L}_n^m} \oplus \left(M_{\mathcal{L}_n^m}\right)^{-1}$ and $g \oplus \left(M_{\mathcal{L}_n^m} \oplus \left(M_{\mathcal{L}_n^m}\right)^{-1}\right)$). Thus, we can solve problem $RR[\mathcal{L}_n]$ in time $O\left(\log^2(m)\right)$ using an oracle for $GC[\mathcal{L}_n]$.

Corollary 3. *If $n \geq 3$ the problem $GC[\mathcal{L}_n]$ is P complete.*

Proof. We have proven that $RR[\mathcal{L}_n]$ is NC reducible to $GC[\mathcal{L}_n]$, whenever $n \geq 2$. Moore and Nilsson proved that if $n \geq 3$, then the problem $RR[\mathcal{L}_n]$ is P complete [8]. Thus, we have that for all $n \geq 3$ the problem $GC[\mathcal{L}_n]$ is P complete

We use the symbols SPP and GC to denote The Sandpile Prediction Problem and The group Computation Problems on general sandpile graphs.

Corollary 4. *SPP is NC reducible to GC.*

Proof. We have proven that for all $n \geq 3$ the problem $GC[\mathcal{L}_n]$ is P complete, it implies that GC is P complete, and it implies that SPP is NC reducible to GC, given that SPP belongs to P.

5 Concluding remarks

There is another way of thinking on our results: our present work can be considered as an instance of *The Average Case Analysis of Simulation Algorithms.* Consider the theorem below. Let $\mathcal{SA}$ be the naive (sequential) sandpile automata simulation algorithm, and let $\mathcal{B}$ be the parallel sandpile automata simulation algorithm (we topple all the unstable nodes at once). We will use the symbol $t_{\mathcal{SA}}(G, f, g)$ to denote the running time of $\mathcal{SA}$ on input (G, f, g) and we will use the symbol $t_{\mathcal{B}}(G, f, g)$ to denote the running time of $\mathcal{B}$ on input (G, f, g).

Theorem 10. *Let $n, m \geq 2$, we have that:*

1. *There exist two positive constants C and D such that for any $n \geq 1$*

$$\Pr_{f,g \in \mathcal{K}(\mathcal{L}_n^m)} \left[t_{\mathcal{SA}}(\mathcal{L}_n^m, f, g) \geq C m^{n+1} \right] \geq D$$

2. *There exist two positive constants C and D such that for any $n \geq 1$*

$$\Pr_{f,g \in \mathcal{K}(\mathcal{L}_n^m)} \left[t_{\mathcal{B}}(\mathcal{L}_n^m, f, g) \geq C m \right] \geq D$$

Proof. We have already proven item 1. We prove item 2. Let C and D be the constants in the statement of item 1 and let $f, g \in \mathcal{K}(\mathcal{L}_n^m)$ be two critical configurations such that $L(f, g) \geq C m^{n+1}$. Then, there exists a node v which is toppled at least Cm times. If we are using the parallel updating protocol (that is, if we are running the algorithm $\mathcal{B}$) the topplings performed at v have to be performed at different times, and it implies that $t_{\mathcal{B}}(\mathcal{L}_n^m, f, g) \geq Cm$. Thus, we have that

$$\Pr_{f,g \in \mathcal{K}(\mathcal{L}_n^m)} \left[t_{\mathcal{B}}(\mathcal{L}_n^m, f, g) \geq C m \right] \geq D$$

Last theorem suggests that $GC[\mathcal{L}_n]$ exhibits some type of average case hardness. We already know that, if $n \geq 3$, the problem $GC[\mathcal{L}_n]$ is P-complete. We believe that for all $n \geq 2$ the problem $GC[\mathcal{L}_n]$ is *strictly $m^{\frac{1}{2}}$-hard on average for P*, it means that, given $\mathcal{A}$ a parallel algorithm solving $GC[\mathcal{L}_n]$, there exists two positive constants C and D such that

$$\Pr_{m \geq 2; f,g \in \mathcal{K}(\mathcal{L}_n^m)} \left[t_{\mathcal{A}}(\mathcal{L}_n^m, f, g) \geq C m^{\frac{n}{2}} \right] \geq D$$

Conjecture 1. Given $n \geq 2$, the problem $GC[\mathcal{L}_n]$ is strictly $m^{\frac{1}{2}}$-hard on average for P (see reference [6]).

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References

[1] L. Babai. The abelian sandpile model. Manuscript, available at http://people.cs.uchicago.edu/~laci/REU05/.

[2] L. Babai, I. Gorodezky. Sandpile Transience on the Grid is Polynomially Bounded. Proc. 2007 ACM-SIAM Symposium on Discrete Algorithms, pgs 627-636.

[3] P. Bak. *How nature works: The Science of Self-organized Criticality.* Copernicus, NY, 1996.

[4] P. Bak, C. Tang, K. Wiesenfeld. Self-organized Criticality. *Physical Review A,* 38:364-374 (1988).

[5] D. Dhar. Theoretical Studies of Self-organized Criticality. *Physica A.* 369:29-70, 2006.

[6] C. Mejia, A. Montoya. On the Complexity of Sandpile Prediction Problems. *Electronic Notes in Theoretical Computer Science.* 252:229-245, 2009.

[7] P. Miltersen. The computational complexity of one-dimensional sandpiles. *Theory of computing systems.* 41(1):119-125, 2007.

[8] C. Moore, M. Nilsson. The computational complexity of sandpiles. *Journal of Statistical Physics.* 96:205-224, 1999.

[9] G. Tardos. Polynomial bound for a chip firing game on graphs. *SIAM J. Discrete Mathematics.* 1:397-398, 1988.

[10] E. Toumpakari. *On the abelian sandpile model.* Ph.D. Thesis, University of Chicago, 2005.