

ON OPEN MAPS BETWEEN DENDRITES

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ABSTRACT. In this paper we use a result by J. Krasinkiewicz to present a description of the topological behavior of an open map defined between dendrites. It is shown that, for every such map $f: X \rightarrow Y$, there exist n subcontinua X_1, X_2, \dots, X_n of X such that $X = X_1 \cup X_2 \cup \dots \cup X_n$, each set $X_i \cap X_j$ consists of at most one element which is a critical point of f , and each map $f|_{X_i}: X_i \rightarrow Y$ is open, onto and can be lifted, in a natural way, to a product space $Z_i \times Y$ for some compact and zero-dimensional space Z_i . We also study the ω -limit sets $\omega(x)$ of a self-homeomorphism $f: X \rightarrow X$ defined on a dendrite X . It is shown that $\omega(x)$ is either a periodic orbit or a Cantor set (and if this is the case, then $f|_{\omega(x)}$ is an adding machine).

1. INTRODUCTION

It is well known that each open map from the interval $[0, 1]$ to itself is an n -fold branched covering map (i.e., there exist $n \in \mathbb{N}$ and n subcontinua X_1, X_2, \dots, X_n of $[0, 1]$ such that $[0, 1] = X_1 \cup X_2 \cup \dots \cup X_n$, each set $X_i \cap X_j$ contains at most one element, for $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$, and each map $f_i = f|_{X_i}: X_i \rightarrow [0, 1]$ is a homeomorphism). Based on this fact, the dynamics of such maps have been extensively investigated (see for example [MT88]). Since every open map of a finite tree, with at least one branch-point, onto itself is a homeomorphism (Theorem 3.1), it is natural to investigate open maps on dendrites. Easy examples show that a straight forward generalization of the above result for the interval is false. In this paper we formulate a correct generalization for the class of dendrites (see Theorem 4.4).

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Dendrites appear naturally as the Julia set of a complex polynomial. If, for example, $p: \mathbb{C} \rightarrow \mathbb{C}$ is the map defined by $p(z) = z^2 + c$, then for certain values of c , the Julia set J of p is a dendrite and the map $p|_J: J \rightarrow J$ is a branched covering [Mil00]. In particular, $p|_J$ is open. The dynamics of such maps is still not well understood (cf. [BL02] and [Thu85]) and serves as a motivation for this paper.

The paper is divided in 5 sections. After the introduction, we write in Section 2 some notions and auxiliary results. Then in Section 3 we present some conditions under which an open map defined between dendrites must be a homeomorphism. In this section we also study the ω -limits sets of a self-homeomorphism $f: X \rightarrow X$ defined on a dendrite X . Later in Section 4 we present a consequence of a theorem by Krasinkiewicz that will allow us to prove the main theorem of the paper (Theorem 4.3). Finally, in Section 5 we collect some other results involving open maps between dendrites.

2. NOTIONS AND AUXILIARY RESULTS

All spaces considered in this paper are assumed to be metric. If X is a space, $p \in X$ and $\epsilon > 0$, then $B_X(p, \epsilon)$ denotes the open ball around p of radius ϵ . If $A \subset X$, then the symbols $\text{cl}_X(A)$, $\text{int}_X(A)$ and $\text{bd}_X(A)$ stands for the closure, the interior and the boundary of A in X , respectively. Moreover, the symbol $|A|$ represents the cardinality of A .

A *continuum* is a nonempty, compact and connected metric space. The topological limit, with respect to the Hausdorff metric, of a sequence of closed nonempty sets $(Y_n)_n$ in a metric space is denoted by $\text{Lim } Y_n$.

A *dendrite* is a locally connected continuum that contains no simple closed curves. For a dendrite X it is known that any subcontinuum of X is a dendrite [Nad92, Corollary 10.6], every connected subset of X is arcwise connected [Nad92, Proposition 10.9], and the intersection of any two connected subsets of X is connected [Nad92, Theorem 10.10]. Given points p and q in a dendrite X , there is only one arc from p to q in X . We denote such an arc by pq .

A *map* is a continuous function. A map f from a continuum X onto a continuum Y is said to be

- *open* if the image of any open subset of X is an open subset of Y ;
- *interior at* $x \in X$ if for every open set U of X such that $x \in U$, we have $f(x) \in \text{int}_Y(f(U))$;
- *confluent* provided that for any subcontinuum Q of Y and any component C of $f^{-1}(Q)$, we have $f(C) = Q$;

- *monotone* if for any $y \in Y$, the set $f^{-1}(y)$ is connected;
- *light* if for any $y \in Y$, the set $f^{-1}(y)$ is zero-dimensional.

It is well known that a map is open if and only if it is interior at each point of its domain. Moreover, any open map is confluent [Nad92, Theorem 13.14]. It is also known that confluent light maps onto a locally connected continuum are open.

For a dendrite X and a point $p \in X$ we denote the *order of p at X* by $\text{ord}_p X$. Points of order 1 in X are called *end-points* of X . The set of all such points is denoted by $E(X)$. It is known that $E(X)$ is zero-dimensional. It is easy to see that if C is a connected subset of X , then the set $C \setminus E(X)$ is arcwise connected. Points of order 2 in X are called *ordinary points* of X . The set of all such points is denoted by $O(X)$. It is known that $O(X)$ is dense in X [Nad92, 10.42]. Points of order greater than 2 are called *branch points* of X . The set of all such points is denoted by $B(X)$. It is known that $B(X)$ is countable [Nad92, Theorem 10.23]. Moreover $\text{ord}_p X \leq \aleph_0$ for any $p \in X$. Note that $X = E(X) \cup O(X) \cup B(X)$.

For a dendrite X and subcontinua A and B of X such that $A \cap B \neq \emptyset$ we define a map $r: A \cup B \rightarrow A$ as follows. If $x \in A$ we put $r(x) = x$ and if $x \in (A \cup B) \setminus A$ then $r(x)$ is the unique point of $A \cap C$ where C is any irreducible arc in $A \cup B$ from x to a point of A . It is known that r is a monotone retraction from $A \cup B$ onto A [Nad92, Lemma 10.25]. The map r is called the *first point map from $A \cup B$ to A* .

If $f: X \rightarrow Y$ is a map then a point $p \in X$ is said to be

- a *fixed point of f* if $f(p) = p$;
- a *periodic point of f* if there exists $n \in \mathbb{N}$ such that $f^n(p) = p$;
- a *critical point of f* if for any neighborhood U of p there exist $x_1, x_2 \in U$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

We denote by $\text{Fix}(f)$, $P(f)$ and \mathfrak{C} the set of fixed, periodic and critical points of f , respectively. It is known that if $f: X \rightarrow X$ is a map and X is a dendrite, then $\text{Fix}(f) \neq \emptyset$ [Why, Corollary 3.21, p. 243].

If X is a space then an arc pq in X is called a *free arc in X* provided that $pq \setminus \{p, q\}$ is open in X . The following theorem collects some results from Section 6 of [CCP94].

Theorem 2.1. *Let $f: X \rightarrow Y$ be an open map from a dendrite X onto a continuum Y . Then*

- (2.1.1) Y is a dendrite;
- (2.1.2) f is light;
- (2.1.3) $\text{ord}_{f(p)} Y \leq \text{ord}_p X$ for any $p \in X$;

- (2.1.4) if $\text{ord}_p X = \aleph_0$, then $\text{ord}_{f(p)} Y = \aleph_0$;
- (2.1.5) $f(E(X)) \subset E(Y)$;
- (2.1.6) $f^{-1}(B(Y)) \subset B(X)$;
- (2.1.7) the set $f^{-1}(y)$ is finite for any $y \in Y \setminus E(Y)$;
- (2.1.8) the set $f^{-1}(E(Y)) \setminus E(X)$ is finite;
- (2.1.9) the image under f of a free arc in X is a free arc in Y ;
- (2.1.10) for each subcontinuum B of Y and for each $p \in f^{-1}(B)$, there is a subcontinuum A of X containing p and such that the map $f|_A : A \rightarrow B$ is a homeomorphism.

The following basic result will be used in Section 4.

Theorem 2.2. *Let X be a dendrite and let M be a subset of X such that $E(X) \subset M$ and $M \setminus E(X)$ is closed in X . Let C be a component of $X \setminus M$. Then C is open and closed in $X \setminus M$.*

PROOF. Note that $X \setminus E(X)$ is connected and locally connected. Hence $X \setminus M$ is locally connected and the required result follows easily. \square

3. HOMEOMORPHISMS AND ω -LIMIT SETS

In this section we provide sufficient conditions which imply that an open map, defined between dendrites, must be a homeomorphism. Later we will study the ω -limit sets of a self-homeomorphism $f: X \rightarrow X$ defined on a dendrite X . We start with a self-open map defined on a non-trivial tree, in which case no additional assumptions are needed, i.e. such a map must be a homeomorphism.

Theorem 3.1. *Let $f: X \rightarrow X$ be an open map from a finite tree X onto itself. If $B(X) \neq \emptyset$, then f is a homeomorphism.*

PROOF. Put $n = |B(X)|$ and let $B(X) = \{b_1, b_2, \dots, b_n\}$. For any given $i \in \{1, 2, \dots, n\}$, let $a_i \in X$ be such that $f(a_i) = b_i$. Put $B = \{a_1, a_2, \dots, a_n\}$. By (2.1.6), $B \subset B(X)$ and since $|B| = n$, it follows that $B(X) = B$. This shows that $f(B(X)) = B(X)$ and $f^{-1}(B(X)) = B(X)$. Hence the map $f|_{B(X)} : B(X) \rightarrow B(X)$ is one-to-one and onto. To finish the proof it suffices to show, by (2.1.5), that $f^{-1}(E(X)) \subset E(X)$.

To see this, suppose there exists $v \in X \setminus E(X)$ such that $w = f(v) \in E(X)$. Since $B(X)$ is finite and $f(B(X)) = B(X)$ it follows that $v \in O(X)$ and there is a connected open subset U of X such that $v \in U \subset O(X)$. Since f is light U can be chosen so that $X \setminus f^{-1}(f(U))$ has at least two components C and D . By (2.1.3) and the inclusion $U \subset O(X)$ we have $f(U) \cap B(X) = \emptyset$ and

$f(U) \cap E(X) = \{w\}$. Thus $X \setminus f(U)$ is a subcontinuum of X that contains $B(X)$. Note that $f^{-1}(X \setminus f(U)) = X \setminus f^{-1}(f(U))$, so both C and D are components of $f^{-1}(X \setminus f(U))$. By the confluence of f we have $f(C) = f(D) = X \setminus f(U)$. The latter contradicts the fact that $f|_{B(X)}$ is one-to-one and completes the proof. \square

In the following theorem we give some conditions under which a confluent map between dendroids must be a homeomorphism. Recall that a *dendroid* is an arc-wise connected continuum such that the intersection of any two of its subcontinua is connected. Note that dendrites are locally connected dendroids. We extend the definition of an end-point in a dendrite as follows. Suppose X is a dendroid. Then a point $e \in X$ is called an *end-point* of X if e is an end-point of every arc in X which contains e . Note that if X is locally connected (and hence if X is a dendrite), this implies that the order of X at e is one. As before we denote the set of all end-points of a dendroid X by $E(X)$.

Theorem 3.2. *Let $f: X \rightarrow Y$ be a map from a dendroid X onto a dendroid Y . Let us assume that:*

(3.2.1) *f is confluent and light,*

(3.2.2) *$f^{-1}(E(Y)) = E(X)$ and the map $f|_{E(X)}: E(X) \rightarrow E(Y)$ is one-to-one.*

Then f is a homeomorphism.

PROOF. Let us assume, on the contrary, that there exist $x, y \in X$ with $x \neq y$ and $f(x) = f(y)$. By (3.2.2) $f(x) \notin E(Y)$ and, by (3.2.1), the set $f^{-1}(f(x))$ is zero-dimensional. Then we can assume, without loss of generality, that $xy \cap f^{-1}(f(x)) = \{x, y\}$. Since $f(x) \notin E(Y)$, and $f(z) \neq f(x)$ for all $z \in xy \setminus \{x, y\}$, there is $e \in E(Y)$ such that $ef(x) \cap f(xy) = \{f(x)\}$. Let C_x and C_y be the components of $f^{-1}(ef(x))$ such that $x \in C_x$ and $y \in C_y$. Since f is confluent, we have $f(C_x) = f(C_y) = ef(x)$. Take points $a \in C_x$ and $b \in C_y$ such that $f(a) = f(b) = e$. By (3.2.2) we have $a, b \in E(X)$ and $a = b$. Then the continuum $C_x \cup xy \cup C_y$ contains a simple closed curve, a contradiction. \square

The following easy corollary will be used in the proof of Theorem 4.4. Another proof can be obtained using the corollary that appears at the end of page 199 of [Why].

Corollary 3.3. *Let $f: X \rightarrow Y$ be an open map from a dendrite X onto a dendrite Y . If f has no critical points, then f is a homeomorphism.*

PROOF. Let f be as assumed. Since f has no critical points, $f^{-1}(E(Y)) \subset E(X)$, and since f is onto and $f(E(X)) \subset E(Y)$ we have $f^{-1}(E(Y)) = E(X)$. This implies that $f|_{E(X)}$ is one-to-one. To see this consider two distinct points

$e_1, e_2 \in E(X)$ such that $f(e_1) = f(e_2)$. Then, since f is light, $f(e_1e_2) = Z$ is a (non-degenerate) continuum. Let $y \in E(Z) \setminus \{f(e_1)\}$ and $x \in e_1e_2 \setminus \{e_1, e_2\}$ such that $f(x) = y$. Then x is a critical point of f , a contradiction. By Theorem 3.2, f is a homeomorphism. \square

Now we turn our attention to self-homeomorphisms defined on a dendrite. The next two results involves the set of fixed points of any such map.

Lemma 3.4. *Let X be a dendrite and $g: X \rightarrow X$ a homeomorphism from X onto itself. Let $a, b \in X$ be such that $a \neq b$ and $g(b) \in X \setminus ab$. Let D be the component of $X \setminus \{b\}$ that contains $g(b)$. Then $\text{Fix}(g) \cap \text{cl}_X(D) \neq \emptyset$.*

PROOF. By a standard construction of a maximal Borsuk ray (see [Hag86]), there is a map $\varphi: [0, \infty) \rightarrow \text{cl}_X(D)$ such that $\varphi(0) = b$, $\varphi(t) \in bg(\varphi(t)) \setminus \{g(\varphi(t))\}$ for every $t \in [0, \infty)$, $\text{cl}_X(\varphi([0, \infty))) \setminus \varphi([0, \infty)) = \{y\}$ and $g(y) = y$. Then $y \in \text{Fix}(g) \cap \text{cl}_X(D)$. \square

Lemma 3.5. *Let X be a dendrite and $g: X \rightarrow X$ a homeomorphism from X onto itself. If $E(X) \cap \text{Fix}(g) \neq \emptyset$, then $|\text{Fix}(g)| \geq 2$.*

PROOF. Let $e \in E(X) \cap \text{Fix}(g)$ and assume that $\text{Fix}(g) = \{e\}$. Let $p \in X \setminus \{e\}$. Note that $C = ep \cap eg(p)$ is an arc that contains e as one end-point. Let v be the other end-point of C . Since $g(e) = e$ and g is a homeomorphism, we have $g(ep) = eg(p)$, so $g(v) \in eg(p)$. Thus either $v \in eg(v) \setminus \{g(v)\}$ or $g(v) \in ev \setminus \{v\}$. Let us assume first that $v \in eg(v) \setminus \{g(v)\}$. Let D be the component of $X \setminus \{v\}$ that contains $g(v)$. By Lemma 3.4, $\text{Fix}(g) \cap \text{cl}_X(D) \neq \emptyset$. Let us assume now that $g(v) \in ev \setminus \{v\}$ and let E be the component of $X \setminus \{g(v)\}$ that contains v . By Lemma 3.4, applied to g^{-1} , we have $\text{Fix}(g^{-1}) \cap \text{cl}_X(E) \neq \emptyset$. In any case we found a fixed point of g different than e . \square

From now on, in this section, $f: X \rightarrow X$ represents a homeomorphism from a dendrite X onto itself. Given $x \in X$ the set $\omega(x)$ of points $y \in X$ such that, for any neighborhood U of y and any $N \in \mathbb{N}$, there is $n > N$ such that $f^n(x) \in U$ is called the ω -limit set of f . Note that $\omega(x) = \limsup f^n(x)$. In this section we will prove that either $\omega(x)$ is a periodic orbit or a Cantor set. To this aim let us consider the collection \mathcal{C} of all components of $X \setminus \text{Fix}(f)$. Since $\text{Fix}(f)$ is a closed subset of the locally connected continuum X , the elements of \mathcal{C} are open subsets of X . Moreover if $C \in \mathcal{C}$, then $C \cap \text{Fix}(f) = \emptyset$ so $\text{cl}_X(C) \cap \text{Fix}(f) \subset E(\text{cl}_X(C))$. In the following lemma we present more properties of \mathcal{C} and its elements.

Lemma 3.6. *The following properties are satisfied:*

- (3.6.1) \mathcal{C} is countable;
 (3.6.2) $f(C) \in \mathcal{C}$ for any $C \in \mathcal{C}$;
 (3.6.3) if $C \in \mathcal{C}$, then $|\text{cl}_X(C) \cap \text{Fix}(f)| \leq 2$;
 (3.6.4) if $C \in \mathcal{C}$ and $|\text{cl}_X(C) \cap \text{Fix}(f)| = 2$, then $f(C) = C$ and if we write $\text{cl}_X(C) \cap \text{Fix}(f) = \{a, b\}$ then for any $x \in C$ either $\omega(x) = \{a\}$ or $\omega(x) = \{b\}$;
 (3.6.5) if $C \in \mathcal{C}$, $|\text{cl}_X(C) \cap \text{Fix}(f)| = 1$ and $f^n(C) \neq C$ for all $n \in \mathbb{N}$, then $\omega(x) = \text{cl}_X(C) \cap \text{Fix}(f)$ for any $x \in C$.

PROOF. Let D be a countable dense subset of X and $C_i \in \mathcal{C}$. Since C_i is open it follows that $C_i \cap D \neq \emptyset$ so we can pick a point $d_i \in C_i \cap D$. Note that if C_i and C_j are different elements of \mathcal{C} , then $d_i \neq d_j$. Thus since D is countable, the collection \mathcal{C} is countable as well. This shows (3.6.1).

To show (3.6.2) let $C \in \mathcal{C}$. Note that $f(\text{Fix}(f)) = \text{Fix}(f)$. Since f is a homeomorphism $f(C)$ is a component of $f(X \setminus \text{Fix}(f)) = f(X) \setminus f(\text{Fix}(f)) = X \setminus \text{Fix}(f)$, so $f(C) \in \mathcal{C}$.

To show (3.6.3) let $C \in \mathcal{C}$ and assume that $|\text{cl}_X(C) \cap \text{Fix}(f)| \geq 3$. Let a, b and c be three different elements of $\text{cl}_X(C) \cap \text{Fix}(f)$. Consider the arcs ab, bc and ac in $\text{cl}_X(C)$ and note that $ab \cap bc \cap ac = \{t\} \subset C$. Since f is a homeomorphism that fixes a, b and c we have $t \in C \cap \text{Fix}(f)$, a contradiction. This shows (3.6.3).

Now assume that $C \in \mathcal{C}$ is such that $|\text{cl}_X(C) \cap \text{Fix}(f)| = 2$. Put $\text{cl}_X(C) \cap \text{Fix}(f) = \{a, b\}$ and take $x \in C$. Let r be the first point map from X to $ab \subset \text{cl}_X(C)$. It is easy to see that

$$1) \quad r(z) = az \cap ab \cap bz, \text{ for any } z \in X.$$

In particular $r(x) = ax \cap ab \cap bx$. Since $a, b \in \text{Fix}(f)$, $f(ab) = ab$, $f(ax) = af(x)$, $f(bx) = bf(x)$, and f is a homeomorphism

$$f(r(x)) = f(ax \cap ab \cap bx) = af(x) \cap ab \cap bf(x).$$

Applying 1) to $z = f(x)$ we have $af(x) \cap ab \cap bf(x) = r(f(x))$. Hence $f(r(x)) = r(f(x))$, so $f^n(r(x)) = r(f^n(x))$ for any $n \in \mathbb{N}$. Note that $r(x) \in ab \setminus \{a, b\}$, so $f(r(x)) \neq r(x)$. This implies that the arcs $xr(x)$ and $f(x)r(f(x))$ are disjoint. Now, since $r(x), f(r(x)) \in ab \setminus \{a, b\}$ and $r(x) \neq f(r(x))$, either $f(r(x)) \in br(x)$ or $f(r(x)) \in ar(x)$. Let us assume, without loss of generality, that $f(r(x)) \in r(x)b$. Then $f|_{ab}$ is a homeomorphism whose graph lays above the diagonal (except at points a and b), so $f^n(z) \rightarrow b$ for any $z \in ab \setminus \{a, b\}$. In particular $f^n(r(x)) \rightarrow b$ and since the arcs in the sequence $(f^n(x)r(f^n(x)))_n$ are mutually disjoint, it follows that $f^n(x) \rightarrow b$. Thus $\omega(x) = \{b\}$. To complete the proof of (3.6.4) we have to see that $f(C) = C$. Let us assume that there is $y \in C$ such that $f(y) \notin C$.

Then $r(f(y)) \in \{a, b\}$, so $r(y)$ is an element of X such that $f(r(y)) = r(f(y)) \in \{a, b\}$, and this contradicts the fact that f is one-to-one. Thus $f(C) \subset C$. By (3.6.2), $C \subset f(C)$, so $f(C) = C$. The proof of (3.6.4) is complete.

To show (3.6.5) let $C \in \mathcal{C}$ be such that $|\text{cl}_X(C) \cap \text{Fix}(f)| = 1$ and $f^n(C) \neq C$, for all $n \in \mathbb{N}$. Put $\text{cl}_X(C) \cap \text{Fix}(f) = \{a\}$ and let $x \in C$. By (3.6.2) and (3.6.4), $(f^n(C))_n$ is a sequence of mutually disjoint elements of \mathcal{C} such that, for any $n \in \mathbb{N}$, $f^n(\text{cl}_X(C)) \cap \text{Fix}(f) = \{a\}$. Since X is locally connected $f^n(\text{cl}_X(C)) \rightarrow \{a\}$, so $\omega(x) = \{a\}$ for any $x \in C$. \square

Let $C \in \mathcal{C}$ be such that $|\text{cl}_X(C) \cap \text{Fix}(f)| = 1$ and $f^n(C) = C$ for some $n \in \mathbb{N}$. Put $\text{cl}_X(C) \cap \text{Fix}(f) = \{a\}$. If $n = 1$ then $f|_{\text{cl}_X(C)}$ is a homeomorphism from the dendrite $\text{cl}_X(C)$ onto itself such that $a \in \text{Fix}(f|_{\text{cl}_X(C)}) \cap E(\text{cl}_X(C))$. Then, by Lemma 3.5, $|\text{cl}_X(C) \cap \text{Fix}(f)| = |\text{Fix}(f|_{\text{cl}_X(C)})| \geq 2$. Since this is a contradiction, we have $n > 1$.

We say that an element $C \in \mathcal{C}$ is an *end-periodic component* of $X \setminus \text{Fix}(f)$ (or simply, that C is end-periodic) if $|\text{cl}_X(C) \cap \text{Fix}(f)| = 1$ and $f^n(C) = C$ for some $n > 1$. By (3.6.2), (3.6.4) and (3.6.5) the image, under f , of an end-periodic component of $X \setminus \text{Fix}(f)$ is an end-periodic component of $X \setminus \text{Fix}(f)$. We say that, for an element $x \in X$, $\omega(x)$ is a *periodic orbit* if there exists $y \in P(f)$ such that $\omega(x) = \{f^n(y) : n \in \mathbb{N} \cup \{0\}\}$. We understand that $f^0(y) = y$ for any $y \in X$.

Let us assume that $x \in X$ is such that $\omega(x)$ is not a periodic orbit. Then if $j \in \mathbb{N}$ we have $x \in X \setminus \text{Fix}(f^j)$. Since f^j is a homeomorphism from X onto itself, the family \mathcal{C}_j of components of $X \setminus \text{Fix}(f^j)$ satisfies properties (3.6.1)-(3.6.5), where \mathcal{C} is replaced by \mathcal{C}_j and f by f^j . Let $C(j-1) \in \mathcal{C}_j$ be such that $x \in C(j-1)$. If $C(j-1)$ is not end-periodic then, by (3.6.4) and (3.6.5), $\omega(x) \in \text{Fix}(f^j)$. Since this contradicts the fact that $\omega(x)$ is not a periodic orbit, $C(j-1)$ is end-periodic. Put $\text{cl}_X(C(j-1)) \cap \text{Fix}(f^j) = \{d(j-1)\}$ and note that $d(j-1)$ is an end-point of $\text{cl}_X(C(j-1))$. Moreover, since $C(j-1)$ is end-periodic, there exists $n_{j-1} > 1$ such that $f^{jn_{j-1}}(C(j-1)) = C(j-1)$. We have shown the following result.

Lemma 3.7. *If $x \in X$ is such that $\omega(x)$ is not a periodic orbit then, for any $j \in \mathbb{N}$, we have $x \in C(j-1)$ where $C(j-1)$ is an end-periodic component of $X \setminus \text{Fix}(f^j)$. Moreover if $\text{cl}_X(C(j-1)) \cap \text{Fix}(f^j) = \{d(j-1)\}$, then $d(j-1)$ is an end-point of $\text{cl}_X(C(j-1))$ and $f^{jn_{j-1}}(C(j-1)) = C(j-1)$ for some integer $n_{j-1} > 1$.*

Let $N = \{n_0, n_1, n_2, \dots\}$ be a sequence of positive integers and let \mathbb{Z}/n_i denote the cyclic group of integers mod (n_i) , with the discrete topology. Then $C_N = \prod_{i=0}^{\infty} \mathbb{Z}/n_i$ is a Cantor set. Define a homeomorphism $h_N : C_N \rightarrow C_N$ by

$h_N(x_0, x_1, \dots) = (y_0, y_1, \dots)$, where y_i is defined as follows. If $x_0 < n_0 - 1$, then $y_0 = x_0 + 1$ and $y_i = x_i$ for all $i > 0$. If there is $j > 0$ such that $x_i = n_i - 1$ for all $i < j$ and $x_j < n_j - 1$, then $y_i = 0$ for all $i < j$, $y_j = x_j + 1$ and $y_l = x_l$ for all $l > j$. If $x_i = n_i - 1$ for all i , then $y_i = 0$ for all i (one can think of $h_N(x_0, x_1, \dots)$ informally as $(x_0, x_1, x_2, \dots) + (1, 0, 0, \dots)$ by adding in each coordinate modulo n_i and carrying). It is not difficult to see that h_N is a minimal homeomorphism. Any homeomorphism $f : C \rightarrow C$ on a Cantor set C for which there exists a sequence of positive integers $N = \{n_0, n_1, \dots\}$ and a homeomorphism $\varphi : C \rightarrow C_N$ such that $f = \varphi^{-1} \circ h_N \circ \varphi$ will be called an *adding machine* (or a *generalized odometer*) [BKP97, D86]. Similarly, given a finite sequence $N(k) = \{n_0, \dots, n_k\}$ of positive integers, we can define a periodic homeomorphism $h_k : \prod_{i=0}^k \mathbb{Z}/n_i \rightarrow \prod_{i=0}^k \mathbb{Z}/n_i$ by restricting h_N to the first $k + 1$ coordinates, where $N(k) \subset N$. Hence, informally, $h_k(x_0, \dots, x_k)$ is defined as $(x_0, x_1, \dots, x_k) + (1, 0, \dots, 0)$ by adding modulo n_i in each coordinate and carrying.

We are ready to prove the above mentioned result about the ω -limit sets of a self homeomorphism defined on a dendrite.

Theorem 3.8. *Let X be a dendrite and $f : X \rightarrow X$ be a homeomorphism from X onto itself. If $x \in X$ then $\omega(x)$ is either a periodic orbit or a Cantor set. Moreover if $\omega(x)$ is a Cantor set, then $f|_{\omega(x)}$ is an adding machine.*

PROOF. Let 0_m and 0_∞ denote the m -tuple of zeros and the infinite sequence of zeros, respectively. Take $x \in X$ and assume that $\omega(x)$ is not a periodic orbit. We will construct a decreasing sequence of subcontinua of X which contain x , as follows. First, by Lemma 3.7, $x \in C(0)$ where $C(0)$ is an end-periodic component of $X \setminus \text{Fix}(f)$. Put $\text{cl}_X(C(0)) \cap \text{Fix}(f) = \{d\}$ and let $n_0 > 1$ be minimal such that $f^{n_0}(C(0)) = C(0)$. Put $D(0) = \text{cl}_X(C(0))$ and note that $D(0) = C(0) \cup \{d\}$ and $f^{n_0}(D(0)) = D(0)$. Put $C(i) = f^i(C(0))$ and $D(i) = f^i(D(0))$ for $1 \leq i < n_0$. Let $N(0) = \{n_0\}$. Since $h_0 : \mathbb{Z}/n_0 \rightarrow \mathbb{Z}/n_0$ is defined as $h_0(m) = m + 1 \pmod{n_0}$, we can also write $D(i) = D(h_0^i(0)) = f^i(D(0))$ for any $0 \leq i < n_0$. Then $C(i)$ is an end-periodic component of $X \setminus \text{Fix}(f)$ and $D(i) \cap \text{Fix}(f) = \{d\}$.

Now define $f_0 = (f^{n_0})|_{D(0)}$ and note that $f_0 : D(0) \rightarrow D(0)$ is a homeomorphism from the dendrite $D(0)$ onto itself. Moreover $\text{Fix}(f_0) \neq \emptyset$ and, by Lemma 3.7, $x \in C(0, 0) = C(0_2)$, where $C(0_2)$ is an end-periodic component of $D(0) \setminus \text{Fix}(f_0)$. Put $D(0_2) = \text{cl}_X(C(0_2))$, $D(0_2) \cap \text{Fix}(f_0) = \{d(0)\}$ and let $n_1 > 1$ be minimal such that $f_0^{n_1}(D(0_2)) = D(0_2)$. Note that $D(0_2) \subsetneq D(0)$ since $d \in D(0) \setminus D(0_2)$. Let $N(1) = \{n_0, n_1\}$. Put $D(h_1^i(0_2)) = f^i(D(0_2))$ for $1 \leq i < n_0 n_1 - 1$, and $d(i) = d(h_0^i(0)) = f^i(d(0))$ for $1 \leq i < n_0 - 1$. Let

$f_1 = (f_0^{n_1})|_{D(0_2)}$ and note that $f_1: D(0_2) \rightarrow D(0_2)$ is a homeomorphism from the dendrite $D(0_2)$ onto itself.

Now we proceed by induction for constructing the subcontinuum $D(0_{j+1})$ from the subcontinuum $D(0_j)$ that contains x . Put $f_{j-1} = (f_{j-2}^{n_{j-1}})|_{D(0_j)}$ and note that $f_{j-1}: D(0_j) \rightarrow D(0_j)$ is a homeomorphism. Hence $\text{Fix}(f_{j-1}) \neq \emptyset$ and, since $\omega(x)$ is not a periodic orbit, $x \in D(0_j) \setminus \text{Fix}(f_{j-1})$. Thus, by Lemma 3.7, x belongs to an end-periodic component $C(0_{j+1})$ of $D(0_j) \setminus \text{Fix}(f_{j-1})$. Put $D(0_{j+1}) = \text{cl}_X(C(0_{j+1}))$, $D(0_{j+1}) \cap \text{Fix}(f_{j-1}) = \{d(0_j)\}$ and let $n_j > 1$ be minimal such that $f_{j-1}^{n_j}(D(0_{j+1})) = D(0_{j+1})$. Let $N(j) = \{n_0, n_1, \dots, n_j\}$. Put $D(h_j^i(0_{j+1})) = f_j^i(D(0_{j+1}))$ for $1 \leq i < n_0 n_1 \cdots n_j - 1$, and $d(h_{j-1}^i(0_j)) = f_j^i(d(0_j))$ for $1 \leq i < n_0 n_1 \cdots n_{j-1} - 1$.

In this way, for $k_i \in \{0, 1, \dots, n_i - 1\}$ and $i \in \{0, 1, \dots, m\}$, we have constructed a subcontinuum $D(k_0, k_1, \dots, k_m)$ of X , such that

$$D(k_0, k_1, \dots, k_m, k_{m+1}) \subsetneq D(k_0, k_1, \dots, k_m)$$

for every $k_{m+1} \in \{0, 1, \dots, n_{m+1} - 1\}$. Define

$$D(k_0, k_1, k_2, \dots) = \bigcap_{m=0}^{\infty} D(k_0, k_1, \dots, k_m)$$

and note that $D(k_0, k_1, k_2, \dots)$ is the intersection of a decreasing sequence of subcontinua of X , thus is a subcontinuum of X as well. Also define

$$d(k_0, k_1, k_2, \dots) = \lim_{m \rightarrow \infty} d(k_0, k_1, \dots, k_m).$$

The limit exists because the sequence of points $(d(k_0, k_1, \dots, k_m))_m$ forms a monotone sequence contained in an arc in X . Define

$$K = \{d(k_0, k_1, k_2, \dots) : k_i \in \{0, 1, \dots, n_i - 1\} \text{ for all } i\}$$

and note that $K \subset X$. Put $N = \{n_0, n_1, n_2, \dots\}$ and $C_N = \prod_i \mathbb{Z}/n_i$. Let $\varphi: K \rightarrow C_N$ be defined by $\varphi(d(k_0, k_1, \dots)) = (k_0, k_1, \dots)$. We claim that φ is a homeomorphism. To see this, let τ be the topology on X and τ_s the topology on K as a subspace of X . If τ_p is the product topology on C_N , then we must show that $\tau_s = \tau_p$. Assume first that U is a basic open set in τ_p . Let $d(k_0, k_1, k_2, \dots) \in U$. Then there is m such that

$$U = \{k_0\} \times \{k_1\} \times \cdots \times \{k_m\} \times \prod_{i>m} \mathbb{Z}/n_i.$$

Let $V = D(k_0, k_1, \dots, k_m) \setminus \{d(k_0, k_1, \dots, k_{m-1})\}$. Note that $d(k_0, k_1, k_2, \dots) \in V \cap K$ and that V is a component of

$$V' = X \setminus \{d, d(k_0), d(k_0, k_1), \dots, d(k_0, k_1, \dots, k_{m-1})\}.$$

Since $V' \in \tau$ and X is locally connected, it follows that $V \in \tau$, so $V \cap K \in \tau_s$. Since $V \cap K \subset U$ it follows that $U \in \tau_s$. This shows that $\tau_p \subset \tau_s$.

To prove the other inclusion let $U \in \tau_s$. Then $U = V \cap K$, for some $V \in \tau$. Let $y = d(k_0, k_1, \dots) \in U$. For simplicity put $D_\infty = D(k_0, k_1, \dots)$ and, for each i , $D_i = D(k_0, k_1, \dots, k_i)$, $d_i = d(k_0, k_1, \dots, k_i)$ and $I_i = D_i \setminus D_\infty$. Then I_i is arcwise connected. To see this we will show that every point $z \in I_i$ can be joined to $d_{i-1} \in I_i$ by an arc lying entirely in I_i . Let zd_{i-1} be the arc in D_i joining z to d_{i-1} . Since y separates d_{i-1} from $D_\infty \setminus \{y\}$, it suffices to show that y does not lie on zd_{i-1} . Note that $d_j \in d_{i-1}y$ for all $j > i-1$. If $y \in zd_{i-1}$, then $d_j \in zd_{i-1}$ for all $j > i-1$. This implies that $z \in D_\infty$, a contradiction. Hence I_i is arcwise connected for all i . Since $(D_i)_i$ is a decreasing sequence it follows that $(I_i)_i$ is a decreasing sequence as well, and since $\bigcap_i I_i = \emptyset$, it follows that $\text{diam}(I_i) \rightarrow 0$. Then there is n such that $I_n \subset V$.

Note that

$$D_n \cap K = \{k_0\} \times \{k_1\} \times \cdots \times \{k_n\} \times \prod_{i>n} \mathbb{Z}/n_i,$$

so $D_n \cap K \in \tau_p$. Moreover $y \in D_n \cap K$ and

$$D_n \cap K = (I_n \cap K) \cup (D_\infty \cap K) \subset (V \cap K) \cup \{y\} = U \cup \{y\} = U.$$

This implies that $U \in \tau_p$ and then $\tau_s \subset \tau_p$. Thus $\tau_s = \tau_p$ and since C_N is a Cantor set in the product topology, K is a Cantor set as well in the subspace topology τ_s .

Since $d(h_j^i(0_{j+1})) = f^i(d(0_{j+1}))$ and

$$d(k_0, k_1, k_2, \dots) = \lim_{m \rightarrow \infty} d(k_0, k_1, k_2, \dots, k_m),$$

it follows that $f(d(k_0, k_1, \dots)) = d(h_N(k_0, k_1, \dots))$. In other words, $f|_K = \varphi^{-1} \circ h_N \circ \varphi$ and $f|_K$ is an adding machine. In particular the orbit of any point in K is dense in K . Now, by [Nad92, Theorem 10.4], $\text{diam}(f^n(D(0_\infty))) \rightarrow 0$ and since $x, d(0_\infty) \in D(0_\infty)$ it follows that $f^n(x) \rightarrow f^n(d(0_\infty))$. Therefore $\omega(x) = \omega(d(0_\infty))$ and since the orbit of $d(0_\infty)$ is dense in K , we have $\omega(d(0_\infty)) = K$. This shows that $\omega(x)$ is a Cantor set and $f|_{\omega(x)}$ is an adding machine. \square

Corollary 3.9. *If $f: X \rightarrow X$ is a homeomorphism from a dendrite X onto itself, then the entropy of f is zero.*

PROOF. Let $h_N: C_N \rightarrow C_N$ be an adding machine. Then h_N is an isometry in the natural metric on C_N and, hence, the entropy of h_N is zero. Moreover, if the entropy of f is positive, then there exists $x \in X$ such that the entropy of $f|_{\omega(x)}$ is positive. Hence the result follows from Theorem 3.8. \square

4. OPEN MAPS BETWEEN DENDRITES

Consider spaces X, Y, M and maps $f: X \rightarrow Y$ and $u: M \rightarrow Y$. Then a map $v: M \rightarrow X$ is said to be a *lifting of u with respect to f* provided that $u = f \circ v$. Denote by $C(X, Y)$ the space of all maps from X into Y . In Section 1 of [K00] the following result is proved.

Theorem 4.1. *Let $f: X \rightarrow Y$ be a confluent and light map from a compact space X onto Y . Let $w: D \rightarrow Y$ be a map from a dendrite D and let $x_0 \in X$ and $\theta \in D$ be such that $f(x_0) = w(\theta)$. Then*

- (4.1.1) *there is a lifting $v: D \rightarrow X$ of w with respect to f such that $v(\theta) = x_0$;*
- (4.1.2) *all liftings of w with respect to f constitute a zero-dimensional compact subset of $C(D, X)$.*

For proving Corollary 4.3 we will use the following reformulation of the conclusion of Theorem 4.1.

Corollary 4.2. *Let $f: X \rightarrow Y$ be a confluent and light map from a compact space X onto Y . Let $w: D \rightarrow Y$ be a map from a dendrite D and let $x_0 \in X$ and $\theta \in D$ be such that $f(x_0) = w(\theta)$. Then there exist a compact zero-dimensional space Z , a point $z_0 \in Z$, and a map $q: Z \times D \rightarrow X$ such that*

- (4.2.1) $q(z_0, \theta) = x_0$,
- (4.2.2) $f(q(z, t)) = w(t)$ for each $(z, t) \in Z \times D$,
- (4.2.3) *for each lifting $\lambda: D \rightarrow X$ of w with respect to f , there is a uniquely determined element $z \in Z$ such that $\lambda(t) = q(z, t)$ for each $t \in D$.*

PROOF. Let Z be the set of all $z \in C(D, X)$ such that z is a lifting of w with respect to f . By (4.1.2) Z is compact and zero-dimensional. Let z_0 be the lifting v guaranteed in (4.1.1) and define $q: Z \times D \rightarrow X$ as $q(z, t) = z(t)$. Then it is easy to show that properties (4.2.1), (4.2.2) and (4.2.3) are satisfied. \square

Corollary 4.3. *Suppose that $f: X \rightarrow Y$ is an open and onto map between dendrites X and Y . Then there is a compact and zero-dimensional set Z and an onto map $q: Z \times Y \rightarrow X$ such that if $\pi_2: Z \times Y \rightarrow Y$ is the map given by $\pi_2(z, y) = y$ for any $(z, y) \in Z \times Y$, then $f \circ q = \pi_2$. Additionally we have the following properties*

- (4.3.1) *if $q(z_1, y_1) = q(z_2, y_2)$, then $y_1 = y_2$;*
- (4.3.2) *if $z \in Z$ and $R = q(\{z\} \times Y)$, then the maps $q|_{\{z\} \times Y}: \{z\} \times Y \rightarrow R$ and $f|_R: R \rightarrow Y$ are homeomorphisms.*

PROOF. Open maps between dendrites are confluent and light, so we can use Corollary 4.2 with the map f as given in the hypothesis, $D = Y$ and w as the

identity map on Y . To show that the map $q: Z \times Y \rightarrow X$ is onto let $x \in X$. By (4.1.1) there is a lifting $\lambda: Y \rightarrow X$ of w with respect to f such that $\lambda(f(x)) = x$. By (4.2.3) there is an element $z \in Z$ such that $\lambda(y) = q(z, y)$ for any $y \in Y$. In particular $q(z, f(x)) = \lambda(f(x)) = x$, so f is onto. Properties (4.3.1) and (4.3.2) are easy to prove. \square

For a natural number n we write $I_n = \{1, 2, \dots, n\}$.

Theorem 4.4. *Let $f: X \rightarrow Y$ be an open map from a dendrite X onto a dendrite Y , and let \mathfrak{C} be the set of critical points of f . Then there exist $n \in \mathbb{N}$ and n subcontinua X_1, X_2, \dots, X_n of X with the following properties:*

- (4.4.1) $X = X_1 \cup X_2 \cup \dots \cup X_n$;
- (4.4.2) for any $i, j \in I_n$ with $i \neq j$ the set $X_i \cap X_j$ contains at most one element.
Moreover if $x \in X_i \cap X_j$ then $x \in \mathfrak{C} \setminus E(X)$ and $f(x) \in E(Y)$;
- (4.4.3) for each $i \in I_n$, the map $f_i = f|_{X_i}: X_i \rightarrow Y$ is open and onto;
- (4.4.4) for each $i \in I_n$, if $f(\mathfrak{C} \cap X_i) \subset E(Y)$, then the map $f_i = f|_{X_i}: X_i \rightarrow Y$ is a homeomorphism;
- (4.4.5) for each $i \in I_n$, if $f(\mathfrak{C} \cap X_i) \setminus E(Y) \neq \emptyset$, it follows that
 - (4.4.5.1) if c is a critical point of f_i and $c \notin E(X_i)$, then $f_i(c) \notin E(Y)$;
 - (4.4.5.2) there is a compact and zero-dimensional set Z_i and an onto map $q_i: Z_i \times Y \rightarrow X$ such that if $\pi_2: Z_i \times Y \rightarrow Y$ is the map given by $\pi_2(z, y) = y$ for any $(z, y) \in Z_i \times Y$, then $f_i \circ q_i = \pi_2$. Additionally we have properties (4.3.1) and (4.3.2) of Corollary 4.3 when Z, X, q and f are replaced by Z_i, X_i, q_i and f_i , respectively.

PROOF. Put $M = f^{-1}(E(Y))$ and consider the sets $O_M = O(X) \cap M$ and $B_M = B(X) \cap M$. Then $M = E(X) \cup O_M \cup B_M$ by (2.1.5). Moreover, the sets $E(X)$, O_M and B_M are pairwise disjoint and, by (2.1.8), the set $M \setminus E(X)$ is finite. Clearly $M \setminus E(X) = O_M \cup B_M$. Now consider the family

$$\mathcal{C} = \{C \subset X: C \text{ is a component of } X \setminus M\}.$$

In the following lines we establish some properties of the family \mathcal{C} .

- 1) If $C \in \mathcal{C}$ then $f(C) = Y \setminus E(Y)$ and $f(\text{cl}_X(C)) = Y$.

To show this let $C \in \mathcal{C}$ and $c \in C$. If $f(c) \in E(Y)$, then $c \in M$, a contradiction to the fact that $C \cap M = \emptyset$. Hence $f(C) \subset Y \setminus E(Y)$. To show the other inclusion fix a point $x \in C$ and let $y \in Y \setminus E(Y)$. Put $z = f(x)$. Note that the set $Y \setminus E(Y)$ is arcwise connected and that $yz \cap E(Y) = \emptyset$. Then for the component K of $f^{-1}(yz)$ that contains x , we have $K \cap M = \emptyset$. Hence $K \subset C$. Since f is confluent

we have $f(K) = yz$, so there is $c \in K$ such that $f(c) = y$. This shows that $Y \setminus E(Y) \subset f(C)$ and the first part of 1) holds. Since f is closed we have

$$f(\text{cl}_X(C)) = \text{cl}_Y(f(C)) = \text{cl}_Y(Y \setminus E(Y)) = Y.$$

Hence 1) holds. Now we claim that

2) If $C, D \in \mathcal{C}$ and $C \neq D$, then $\text{cl}_X(C) \cap D = \emptyset$.

To show this let $C, D \in \mathcal{C}$ be such that $C \neq D$. Note that M is a subset of X such that $E(X) \subset M$ and $M \setminus E(X)$ is finite. Then, by Theorem 2.2, C is open and closed in $X \setminus M$. Thus $\text{cl}_{X \setminus M}(C) \cap D = C \cap D = \emptyset$, so

$$\emptyset = \text{cl}_{X \setminus M}(C) \cap D = \text{cl}_X(C) \cap (X \setminus M) \cap D = \text{cl}_X(C) \cap D.$$

This shows 2).

3) If $C \in \mathcal{C}$, then $\text{cl}_X(C) \setminus C \subset M$.

To see this let $C \in \mathcal{C}$ and take a point $x \in \text{cl}_X(C) \setminus C$. If $x \notin M$, then $x \in D$ for some $D \in \mathcal{C}$. Note that $\text{cl}_X(C) \cap D \neq \emptyset$ and $D \neq C$. This contradicts 2), so 3) holds.

4) If $C, D \in \mathcal{C}$, $C \neq D$ and $B = \text{cl}_X(C) \cap \text{cl}_X(D)$, then either $B = \emptyset$ or B is a one-point set and $B \subset O_M \cup B_M$.

To show this let C, D and B be as assumed. Consider that B is nonempty. Then B is a subcontinuum of X , so $f(B)$ is a subcontinuum of Y . Let us assume that there is a point $b \in B \setminus M$. Then, by 3), $b \in \text{cl}_X(C) \setminus M \subset C$ and $b \in \text{cl}_X(D) \setminus M \subset D$. This implies that $C = D$, which is a contradiction. Hence $B \subset M$. Thus $f(B) \subset f(M) \subset E(Y)$. Since $E(Y)$ is zero-dimensional and $f(B)$ is connected, it follows that $f(B)$ is a one-point set. Hence, by (2.1.2), B is a one-point set too.

Put $B = \{x\}$ and note that $x \in M$. Then $x \in E(X) \cup O_M \cup B_M$. Let us assume that $x \in E(X)$. Fix points $c \in C$ and $d \in D$ and consider the arcs $cx \subset \text{cl}_X(C)$ and $dx \subset \text{cl}_X(D)$. Then $cx \cap dx \subset B = \{x\}$, so the set $cx \cup dx$ is an arc in X with end-points c and d . Since $x \in E(X)$ either $x = c$ or $x = d$. Hence either $\text{cl}_X(C) \cap D \neq \emptyset$ or $\text{cl}_X(D) \cap C \neq \emptyset$. In any situation we contradict assertion 2), so 4) holds.

5) If $C \in \mathcal{C}$, then $E(\text{cl}_X(C)) = \text{cl}_X(C) \cap M$.

To show this note first that $\text{cl}_X(C) \cap M \subset \text{cl}_X(C) \setminus C \subset E(\text{cl}_X(C))$. On the other hand suppose $x \in E(\text{cl}_X(C))$ and $x \notin M$. Then $x \notin E(X)$ so $X \setminus \{x\}$ has at least two components A and B . Assume, without loss of generality, that $\text{cl}_X(C) \setminus \{x\} \subset A$. Choose $a \in A \setminus E(X)$ and $b \in B \setminus E(X)$, then $x \in ab$ and $ab \cap E(X) = \emptyset$. By (2.1.8), $ab \cap M$ is finite and there exists an open sub-arc pq

of ab which contains x such that $pq \cap M = \emptyset$. Then $pq \subset C$ which contradicts the assumption that $x \in E(\text{cl}_X(C))$, and 5) holds.

6) The family \mathcal{C} is finite.

To see this fix a point $y \in Y \setminus E(Y)$. By 1), $f^{-1}(y) \cap C \neq \emptyset$ for each $C \in \mathcal{C}$. By (2.1.7), $f^{-1}(y)$ is finite. Hence \mathcal{C} is finite and 6) holds.

By 6) there exists $n \in \mathbb{N}$ such that $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ and $C_i \neq C_j$ for every $i, j \in I_n$ with $i \neq j$. Given $i \in I_n$ put $X_i = \text{cl}_X(C_i)$. Clearly X_i is a subcontinuum of X . Moreover if $i, j \in I_n$ and $i \neq j$ then, by 4), the set $X_i \cap X_j$ is either empty or it is a one-point set whose only element belongs to $O_M \cup B_M$. By 5) we have $E(X_i) = X_i \cap M$ for any $i \in I_n$. We claim that

7) $X = X_1 \cup X_2 \cup \dots \cup X_n$.

To see this put $X_0 = X_1 \cup X_2 \cup \dots \cup X_n$ and note that $X \setminus M \subset X_0$. Suppose that there is a point $x \in M \setminus X_0$. Then $X \setminus X_0$ is an open subset of X that contains x . Then $f(X \setminus X_0)$ is an open subset of Y that contains $f(x) \in E(Y)$. Hence there exists $y \in f(X \setminus X_0)$ such that $y \notin E(Y)$. Let $a \in X \setminus X_0$ be such that $f(a) = y$. Note that $a \in X \setminus M$ so $a \in X_0$. This contradiction shows that $M \subset X_0$, so 7) holds.

By 7) assertion (4.4.1) holds. Assertion (4.4.2) follows from 1), 3) and 4). To show (4.4.3) let i and f_i be as assumed. By 1) $f_i(X_i) = f(\text{cl}_X(C_i)) = Y$ so f_i is onto. Since f is open, f_i is interior at any point of $X_i \setminus \bigcup_{j \neq i} X_j$. Hence to show that f_i is open it suffices to prove that

8) f_i is interior at any point of $X_i \cap X_j$ for $j \neq i$.

To show this let $j \neq i$ and take a point $x \in X_i \cap X_j$. By 4) $x \in O_M \cup B_M$. Since $O_M \cup B_M$ is finite, there exists an open and connected subset V of X_i such that $V \cap (O_M \cup B_M) = \{x\}$. Note that f_i is interior at any point of $V \setminus \{x\}$. We claim that $y = f(x) \in \text{int}_Y(f(V))$. For suppose that there exists $y_n \in Y \setminus f(V)$ such that $y_n \rightarrow y$. Then $\text{Lim } y_n y = \{y\}$. Since $\dim f^{-1}(y) = 0$, there exists $a \in V \setminus f^{-1}(y)$. Then $ax \subset V$ and $f(ax) \subset f(V)$. Since $y \in E(Y)$ and $f(ax) \subset f(V)$ is a subcontinuum of Y containing y , there exist a first point w_n of $y_n y$ (from y_n) such that $w_n \in f(\text{cl}_X(V))$ and a first point z_n of $w_n y$ (from w_n) such that $z_n \in f(ax)$. Choose $v_n \in ax$ such that $f(v_n) = z_n$ and let K_n be the component of $f^{-1}(w_n z_n)$ containing v_n . Then $f(K_n) = w_n z_n$ and since $\dim f^{-1}(y) = 0$ we have $\text{Lim } K_n = \{x\}$. Hence $K_n \subset V$ for sufficiently large n . Choose n such that $K_n \subset V$ and let $u_n \in K_n$ be such that $f(u_n) = w_n$. Since w_n is the first point of $y_n y \cap f(\text{cl}_X(V))$ we have $w_n \notin \text{int}_Y(f(V))$, contradicting the fact that f is interior at u_n . This completes the proof of 8) and, hence, (4.4.3) holds.

To show assertion (4.4.4) let $i \in I_n$ and assume that $f(\mathfrak{C} \cap X_i) \subset E(Y)$. Then f_i is an open and onto map with no critical points. By Corollary 3.3, f_i is a homeomorphism.

To show assertion (4.4.5) of the theorem, let $i \in I_n$ and assume that $f(\mathfrak{C} \cap X_i) \setminus E(Y) \neq \emptyset$. Let c be a critical point of f_i such that $c \notin E(X_i)$. If $f_i(c) \in E(Y)$, then $c \in X_i \cap M = E(X_i)$ according to 5). This contradiction shows that $f_i(c) \notin E(Y)$, so (4.4.5.1) holds. Finally (4.4.5.2) follows from Corollary 4.3. \square

Corollary 4.5. *Suppose that $f : X \rightarrow Y$ is an open map from the dendrite X onto the dendrite Y such that $f(\mathfrak{C}) \subset E(Y)$, where \mathfrak{C} is the set of critical points of f . Then there exist $n \in \mathbb{N}$ and n subcontinua X_1, \dots, X_n such that $X = \bigcup_{i=1}^n X_i$, $X_i \cap X_j$ is at most one critical point of f and for each $i \in I_n$, $f|_{X_i} : X_i \rightarrow Y$ is a homeomorphism.*

5. OPEN MAPS ON DENDRITES

It is easy to see that the set of critical points \mathfrak{C} of an open map $f : X \rightarrow Y$ between two dendrites can be uncountable. In this section we will show that for an arc $A \subset X$, the set of critical points of the restricted map $f|_A$ is finite. We always assume that $f : X \rightarrow Y$ is an open map from a dendrite X onto a dendrite Y .

Theorem 5.1. *Let A be a subcontinuum of X such that $f|_A$ is one-to-one. Then there is a subcontinuum B of X such that $A \subset B$ and $f|_B : B \rightarrow Y$ is a homeomorphism.*

PROOF. Let C be a component of $Y \setminus f(A)$. By [Nad92, Theorem 5.6] $\text{cl}_Y(C) \cap f(A) \neq \emptyset$. Let $a_C \in A$ be such that $f(a_C) \in \text{cl}_Y(C)$. Since X contains no simple closed curves, we have $\text{cl}_Y(C) \cap f(A) = \{f(a_C)\}$. Moreover, by (2.1.10), there is a subcontinuum A_C of X such that $a_C \in A_C$ and $f|_{A_C} : A_C \rightarrow \text{cl}_Y(C)$ is a homeomorphism. Then

$$B = A \cup \left(\bigcup \{A_C : C \text{ is a component of } Y \setminus f(A)\} \right)$$

satisfies the required conditions. \square

In the next theorem we show that on a given arc $A \subset X$, the map $f|_A$ has only finitely many critical points.

Theorem 5.2. *Let A be an arc in X from a point $a \in X$ to a point $b \in X$. Order A by \leq in such a way that $a \leq b$. Then there are $a = a_0 < a_1 < \dots < a_k = b$ such that $f|_{[a_i, a_{i+1}]}$ is one-to-one, for any $i \in \{0, 1, \dots, k-1\}$ and the set of critical points of $f|_A$ is $\{a_1, a_2, \dots, a_{k-1}\}$.*

PROOF. First assume that $A \setminus \{a, b\}$ contains infinitely many critical points a_i of the map $f|_A$. Given $i \in \mathbb{N}$ note that f is not one-to-one in any neighborhood of a_i in A . By compactness of A it follows that there is a subarc B of A such that B contains infinitely many a_n and $f(B) \neq Y$.

Since we can replace A by B we may assume that $f(A) \neq Y$. Fix an ordinary point $y \in Y \setminus f(A)$. Given $i \in \mathbb{N}$, by (2.1.10), there is a subcontinuum A_i of X such that $a_i \in A_i$ and $f|_{A_i} : A_i \rightarrow Y$ is a homeomorphism. Consider the first point map $r : X \rightarrow A$ from X to A and note that f is one-to-one in the arc xa_i , for any $x \in f^{-1}(y) \cap A_i$. By (2.1.7) the set $f^{-1}(y)$ is finite. Moreover $f^{-1}(y) \cap A = \emptyset$. Put $f^{-1}(y) = \{x_1, x_2, \dots, x_n\}$ and note that $f^{-1}(y) \cap A_i \neq \emptyset$ for every $i \in \mathbb{N}$. Since A contains infinitely many a_i , there exist $s \in \{1, 2, \dots, n\}$ and $N \subset \mathbb{N}$ infinite such that $x_s \in A_i$ for any $i \in N$. Put $c = r(x_s)$. Since X is uniquely arcwise connected and N is infinite, there must exist $i, j \in N$ such that either $a_i < a_j < c$ or $c < a_j < a_i$. Hence $a_j \in ca_i \subset A_i$, contradicting that $f|_{A_i}$ is a homeomorphism. This contradiction shows that $f|_A$ has finitely many critical points a_1, a_2, \dots, a_{k-1} .

Put $a_0 = a$, $a_k = b$ and assume, without loss of generality, that $a_0 \leq a_1 \leq \dots \leq a_{k-1} \leq a_k$ and that the set $A \setminus \{a_0, a_1, \dots, a_k\}$ contains no critical points of $f|_A$. Given $i \in \{0, 1, \dots, k-1\}$ suppose that $f|_{a_i a_{i+1}}$ is not one-to-one. Then there exist $p, q \in a_i a_{i+1}$ such that $f(p) = f(q)$. We can assume that $a_i \leq p < q \leq a_{i+1}$. By (2.1.2) $f(pq)$ is a non-degenerate subcontinuum of Y , so we can take an endpoint y_0 of $f(pq)$ different than $f(p)$. Let $x_0 \in pq$ be such that $f(x_0) = y_0$. Note that $x_0 \in A \setminus \{a_0, a_1, \dots, a_{k+1}\}$. Moreover, it is not difficult to see that x_0 is a critical point of $f|_A$. This contradiction shows that $f|_{a_i a_{i+1}}$ is one-to-one. \square

Remark. Note that Theorem 5.2 does not state that $\mathfrak{C} \cap A$ is finite, where \mathfrak{C} denotes the set of critical points of f . Indeed, easy examples show that this may not be true.

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