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Topological automorphisms of modified Sierpiński gaskets realize arbitrary finite permutation groups

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Abstract

The n-dimensional Sierpiński gasket X, spanned by $n+1$ vertices, has $(n+1)!$ symmetries acting as the symmetric group on the vertices. The object of this note is the remarkable observation that for $n \geqslant 2$ every topological automorphism of X is one of these symmetries. A modification of the arguments yields that, given any finite permutation group $G \leqslant S_{n+1}$ acting on an $(n+1)$-element set, there is a finite subset $T \subseteq X$ such that G is the group of topological automorphisms of $X \setminus T$ considered as a group acting faithfully on the vertices.

Keywords: Sierpiński gasket; Groups of topological automorphisms

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1. Introduction

The Sierpiński gasket is a well-known geometric object which can be obtained in the following way. Start with a (closed) triangle $\Delta = \Delta_0 = U_0$ with the vertex points p_0, p_1 and p_2. Consider the bisection points of the edges which form another (open) triangle Δ_1. Remove Δ_1 from Δ to get the union U_1 of three smaller triangles, each two of them having one point in common. Continue with each of the remaining triangles in the same way to get a union U_2 of 9 triangles of the next smaller generation and so on. The 2-dimensional Sierpiński gasket is the intersection $X = \bigcap_{n=0,1,\ldots} U_n$ of the generated decreasing sequence of sets. Each $x \in X$ is determined by a decreasing sequence of triangles having exactly x in common. In each generation of triangles there are three possibilities. Thus x can be represented by a sequence on the set $3 = \{0, 1, 2\}$. If we take

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the Tychonoff topology on the space 3^{ω} and then take the quotient topology identifying sequences which represent the same point we get a description of the space X. A standard reference for fractal sets like X is [2]. More special and recent results and references on the Sierpiński gasket can be found for instance in [3].

To be more precise and more general than in the first paragraph, start instead of $3 = \{0, 1, 2\}$ with an arbitrary set M equipped with the discrete topology and consider the space M^{ω} of sequences $\vec{x}, \vec{y}, \ldots$ on M with the product topology. Call $\vec{x} = (x_0, x_1, \ldots)$ and $\vec{y} = (y_0, y_1, \ldots)$ equivalent (in symbols $x \sim y$) if there is an n such that $x_k = y_k$ holds for all $k < n$ and all $k > n$ satisfy $x_k = y_n$ and $y_k = x_n$. $\sim$ is an equivalence and we define $X = X^{(M)} = M^{\omega}/{\sim}$ to be the quotient space.

A topological automorphism of the space X is a homeomorphism $f : X \to X$. All topological automorphisms form, with the composition of maps as operation, the group $\mathrm{Aut}(X)$. The group operation is the composition of maps. If $\pi \in S_M$ is any permutation on M let f_π be the map

$$f_\pi : X \to X, \quad (x_n)_{n\in\mathbb{N}} \mapsto \big(\pi(x_n)\big)_{n\in\mathbb{N}}.$$

By the obvious geometric interpretation we call it the symmetry induced by π. $\mathrm{Sym}(X)$ denotes the the group of symmetries on X. Of course $\mathrm{Sym}(X) \subseteq \mathrm{Aut}(X)$.

In this note we prove the converse: For finite M every $f \in \mathrm{Aut}(X)$ is a symmetry (Theorems 1 and 2). Thus $\mathrm{Aut}(X)$ acts faithfully as the symmetric group S_V on the set V of vertices and can be identified with the symmetric group on M. This means that the restriction mapping $\varphi_V : \mathrm{Aut}(X) \to S_V$, $f \mapsto f|_V$, is a well-defined isomorphism.

A modification of the arguments yields Theorem 3, showing that any permutation group acting on a finite set M can be identified with $\mathrm{Aut}(X \setminus T)$, where T is an appropriate finite subset of X.

Problem. Are these results also true for infinite M?

The reader who is interested in a general and recent textbook on permutation groups may confer to [1].

2. Notations, general agreements and idea of the proof

The fixed set M is, for the rest of the paper, supposed to be finite with $|M| \geqslant 3$. It is convenient to introduce several abbreviations. We call constant sequences $\vec{v} = (m, m, \ldots)$, $m \in M$, vertices and write $\vec{v} = (m)$. V denotes the set of all vertices.

Let $\vec{x} = (x_n)_{n\in\mathbb{N}} \sim \vec{y} = (y_n)_{n\in\mathbb{N}}$ with $x_n = y_n$ for all $n < k$, $x_k \neq y_k$ and $x_n = y_k$, $y_n = x_k$ for all $n > k$. Then we write $(x_0, x_1, \ldots, x_{k-1}, x_k/y_k)$ for the equivalence class consisting of $\vec{x}$ and $\vec{y}$ and call it a dyadic point of the kth generation. D_k denotes the set of all dyadic points of the kth generation. The whole set of dyadic points is the union $D = \bigcup_{k=0}^{\infty} D_k$. D is dense in X. Hence every $f \in \mathrm{Aut}(X)$ is determined by its values on D. Let us call the remaining points $\vec{x} \in G = X \setminus V \setminus D$ general points of X. Note that the representation of vertices and general points as a sequence is unique and the identification via $\sim$ matters exactly the dyadic points. For technical convenience we make a further distinction. If in the sequence $\vec{x} = (x_n)_{n\in\mathbb{N}}$ of the general point $\vec{x}$ every $m \in M$ occurs infinitely many times, i.e., for each $m \in M$ the equation $x_n = m$ holds for infinitely many $n \in \mathbb{N}$, we call $\vec{x}$ a generic point. Let us denote the set of generic points by G_0. Most points are generic, as well in the sense of probability theory as well as in that of Baire categories. (We do not need these fact.)

Notations. If $Y \subseteq X$ then $Y(x_0, \ldots, x_n)$ denotes the set of those $\vec{y} = (y_n)_{n\in\mathbb{N}} \in Y$ with $(y_0, \ldots, y_n) = (x_0, \ldots, x_n)$. For fixed $m \in M$, the shift operator $\tau : X(m) \to X$, $(x_0, x_1, x_2, \ldots) \mapsto (x_1, x_2, x_3, \ldots)$ is a well-defined homeomorphism, similarly $\tau^n : X(m_0, m_1, \ldots, m_{n-1}) \to X$ for fixed $(m_0, \ldots, m_{n-1})$. Note that G_0 is shift invariant in the strong sense that $\tau^{-1}(G_0) = G_0 = \tau(G_0)$, which is not the case for V and D.

The subspaces $X' = X \setminus T$ with finite $T \subseteq X$ will play an important role. Therefore we introduce the following manners of speaking. If $T \subseteq Y$ we call X' a Y-mSg (Y-modified Sierpiński gasket). Similarly we call it a Y_1Y_2-mSg if $T \subseteq Y_1 \cup Y_2$, etc. In our proofs VG_0-mSg will occur very often. Of special technical importance for us are, for $Y \subseteq X$, the sets $Y \setminus T^{(m,k)},\ m \in M,\ k \in \mathbb{N}$, with

$$T^{(m,k)} = \bigl\{(x_0 = m, x_1 = m, \ldots, x_{k-1} = m, m/m') \mid m' \neq m\bigr\} \subseteq D_k.$$

The main idea of the proof of our results is that the sets $T = T^{(m,k)}$ are characterized by the purely topological property that no other sets of cardinality $|M| - 1$ make $X' \setminus T$ split into two components of connectedness. Among the points of these sets those of D_0 are the only ones with the further property to be contained in more than one $T^{(m,k)}$. Thus $f(D_0) = D_0$ for every $f \in \mathrm{Aut}(X')$. It follows that the sets $X'(m) \setminus D_0$, $m \in M$, being just the components of $X' \setminus D_0$ have to be permuted by f. The induced permutation $\pi \in S_M$ easily turns out to satisfy $f = f_\pi$. These arguments are carried out in Section 4 after in Section 3 several auxiliary results on the connectedness of the involved sets are presented.

3. Several connectedness properties

For this section let $T \subseteq G_0 \cup V$ be a finite set of generic points and vertices (cf. Section 2). For Theorem 2 we need statements about $X' = X \setminus T$.

Lemma 1. *Every VG_0-mSg $X' = X \setminus T$ is connected, provided $|M| \geqslant 3$.*

Proof. Fix any $\vec{x}_0 \in D_0$. Let C be its component of connectedness (with respect to X'). Since $T \subseteq V \cup G_0$, all (connected) straight lines between points $\vec{x} \in D_0$ are contained in X', hence also in C. (Here we used that for $|M| \geqslant 3$ the set G_0 does not contain points on line segments joining points of D.) In particular we have $\vec{x} \in C$ for all $\vec{x} = (m_0, m_1/m_2) \in D_1$ with pairwise different m_0, m_1, m_2. An arbitrary $\vec{x} = (m, m_1/m_2) \in D_1$ can be joined with at least one of these points within X', hence $D_1 \subseteq C$. Continuing this argument we get $D_k \subseteq C$ for all k, i.e., $D \subseteq C$. Since components are closed and D is dense in X' we get $X' = \overline{D} \subseteq C \subseteq X'$, proving that X' is connected. $\square$

Let us say that Γ is a graph on a set A (undirected, without one point circles) if all $E \in \Gamma$ (called edges) are two element subsets $E = \{a_1 \neq a_2\} \subseteq A$ of A. $\Gamma(A)$ denotes the complete graph on A, i.e., $\Gamma(A) = \{E \subseteq A \mid |E| = 2\}$. The following two lemmas contain several useful and well-known facts concerning connectedness of graphs and topological spaces.

Lemma 2. *Let $\Gamma \subseteq \Gamma(M)$ be a graph on M and $\Gamma_0 = \Gamma(M) \setminus \Gamma$.*

(1) *If $|\Gamma_0| \leqslant |M| - 2$ then Γ is a connected graph.*

(2) *If $|\Gamma_0| = |M| - 1$ then Γ is not connected only if $\Gamma_0 = \{\{m, m'\} \mid m' \neq m\}$ for some $m \in M$.*

Proof. Suppose that Γ is not connected. Then M can be represented as the union of two nonempty disjoint subsets M_1, M_2 such that there are no edges $\{m_1, m_2\}$ in Γ with $m_1 \in M_1$, $m_2 \in M_2$. Let, without loss of generality, $k = |M_1| \leqslant \frac{1}{2}|M| \leqslant |M_2|$. This means that $|\Gamma_0| \geqslant |M_1||M_2| = k(|M| - k)$. If $|\Gamma_0| \leqslant |M| - 2$ this is impossible, yielding the first assertion. If $|\Gamma_0| = |M| - 1$ we conclude $k = 1$, and the second assertion follows. □

Lemma 3. *Let $A = \bigcup_{i \in I} A_i$ be a topological space. Suppose that*

$$\Gamma = \left\{(i_1, i_2) \mid i_1 \neq i_2,\ A_{i_1} \cap A_{i_2} \neq \emptyset\right\}$$

is a connected graph on I and each A_i is connected. Then A is connected.

Proof. Every decomposition of A into components would induce a decomposition of at least one A_i into components, contradiction. □

Lemma 4. *Let X' be a VG_0-mSg, $D' \subseteq D$, D' finite and $X'' = X' \setminus D'$.*

(1) *If $|D'| \leqslant |M| - 2$ then X'' is connected.*

(2) *If $|D'| = |M| - 1$ then X'' is not connected if and only if $D' = T^{(m,k)}$ for some $m \in M$ and $k \in \mathbb{N}$.*

Proof. (1) D' is finite, hence there is a maximal k such that $D' \cap D_k \neq \emptyset$. Then all $X''(m_0, \ldots, m_k)$, $m_i \in M$, are VG_0-mSg and therefore, by Lemma 1, connected. Consider the representation

$$X''(m_0, \ldots, m_{k-1}) = \bigcup_{m' \in M} X''(m_0, \ldots, m_{k-1}, m')$$

and the graph Γ corresponding to it as in Lemma 3. Since $|D'| \leqslant |M| - 2$, Lemma 2 implies that Γ is connected. Hence an application of Lemma 3 shows that all $X''(m_0, \ldots, m_{k-1})$ are connected. Continuing in the same way we get connectedness of all $X''(m_0, \ldots, m_{k-2})$, $X''(m_0, \ldots, m_{k-3})$ etc. and finally of X''.

(2) It is clear that X'' is not connected if $D' = T^{(m,k)}$. Thus it suffices to prove the converse. In order to do this suppose that X'' is not connected. First we show that $D' \subseteq X(m)$ for some $m \in M$. If this would fail, every $X''(m)$ would contain at most $|M| - 2$ points of D', hence be connected by the first part of the lemma. By the same

argument as in the proof of the first part we would get the contradiction that X'' is connected. Fix the (unique) m with $D' \subseteq X'(m)$. Take the maximal value of k such that $D' \subseteq X'_k = X'\ (x_0 = m, \ldots, x_k = m)$. Since k is maximal, at least one point of $X'_k \cap D'$ is in $T^{(m,k)}$. Thus $X''_k = X'_k \setminus D'$ can be represented as a VG_0-mSg with at most $|M| - 2$ deleted points, hence is connected, again by the first part. Obviously $Y = (X'' \setminus X'_k) \cup (T^{(m,k)} \setminus D')$ is connected. (For a formal proof use Lemma 3 where the A_i are connected sets of the form $X''(m_0, \ldots, m_k)$.) Since $X'' = Y \cup X''_k$, this together with Lemma 3 implies that $Y \cap X''_k$ is empty. But this is possible only if $D' \supseteq T^{(m,k)}$. Since $|T^{(m,k)}| = |M| - 1 = |D'|$, both sets coincide. □

4. Proof of the theorems

Consider all subsets A of a VG_0-mSg X' with cardinality $|A| = |M| - 1$ such that $X'' = X' \setminus A$ splits into two components. By Lemma 4 these are exactly the sets $A = T^{(m,k)}$, $m \in M$ and $k \in \mathbb{N}$. If $f \in \mathrm{Aut}(X')$ and A has this property then also $f(A)$ has this property. If $k \geqslant 1$ then $A = T^{(m,k)}$ has an empty intersection with all other sets of this type. On the other hand, for $m_1 \neq m_2$, we have

$$T^{(m_1,0)} \cap T^{(m_2,0)} = \big\{(m_1/m_2)\big\} \neq \emptyset.$$

Thus the points in D_0 are characterized by a topological property which has to be preserved by every $f \in \mathrm{Aut}(X')$, i.e., $f(D_0) = D_0$. $X' \setminus D_0$ splits into the connected components $X'(m)$, $m \in M$. Hence the same must hold for $f(X' \setminus D_0)$, implying $f(X'(m)) = X'(\pi(m))$ for a unique $\pi \in S_M$. Call this π the permutation induced by f. Now we are able to prove our main results.

Theorem 1. *Let $X' \subseteq X$ be a VG_0-mSg, i.e., X' is a Sierpiński gasket minus a finite set of vertices and generic points, and $f \in \mathrm{Aut}(X')$. Then $f(\vec{x}) = f_\pi(\vec{x})$ for all $\vec{x} \in X'$, where $\pi \in S_M$ is the permutation induced by f.*

Proof. By induction on k we will show that

$$f\big(X'(m_0, \ldots, m_k)\big) = X'\big(\pi(m_0), \ldots, \pi(m_k)\big).$$

This implies $f = f_\pi$. If $k = 0$ the assertion is just what we mentioned above. Suppose that the assertion holds for some k. This implies that intersections of the sets $X'(m_0, \ldots, m_k)$ are preserved, i.e.,

$$f(m_0, \ldots, m_k/m'_k) = \big(\pi(m_0), \ldots, \pi(m_k)/\pi(m'_k)\big).$$

Consider the sets $X'_m = X'(m_0, \ldots, m_k, m)$ for fixed $m_0, \ldots, m_k$. By the same argument as above these sets are permuted by f. But this permutation is compatible with the dyadic points only if it is again π. □

Theorem 2. $\mathrm{Aut}(X) = \mathrm{Sym}(X) \cong S_M$ *acts faithfully on V which can be identified with M via $(m) \mapsto m$. This action is given by the restriction mapping $f \mapsto f|_V$.*

Proof. Follows immediately from Theorem 1 and $\mathrm{Sym}(X) \subseteq \mathrm{Aut}(X)$. $\square$

Theorem 3. *For each permutation group $G \leqslant S_M$ acting on the finite set M there is a finite set $T \subseteq X = X^{(M)}$ such that* $\mathrm{Aut}(X')$, $X' = X \setminus T$, *is given by*

$$\mathrm{Aut}(X') = \{ f_\pi|_{X'} \mid \pi \in G \}.$$

Thus $\mathrm{Aut}(X')$ *is isomorphic to G and has the same faithful action on M if we identify V with M via $(m) \mapsto m$.*

Proof. Fix any point $(x_n)_{n\in\mathbb{N}}$ from G_0 and consider its orbit

$$T = \{ (\pi(x_n))_{n\in\mathbb{N}} \mid \pi \in G \}$$

induced by G. Then $X' = X \setminus T$ is a G_0-mSg and $f_\pi(X') = X'$ for all $\pi \in G$. This shows $\supseteq$ in the assertion of the Theorem. The other set-theoretic inclusion follows from Theorem 1. $\square$

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