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A topological characterization of the Sierpiński triangle ☆

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ABSTRACT

We present a topological characterization of the Sierpiński triangle. This answers question 58 from the Problem book of the Open Problem Seminar held at Charles University. In fact we give a characterization of the Apollonian gasket first. Consequently we show that any subcontinuum of the Apollonian gasket, whose boundary consists of three points, is homeomorphic to the Sierpiński triangle.

1. Introduction

A *continuum* means a non-empty compact connected metrizable space. A point x of a space X is called a *local cut-point* if there is a connected open neighborhood U of x such that $U \setminus \{x\}$ is not connected. A *simple closed curve* is any space homeomorphic to the unit circle. An *arc* is any space which is homeomorphic to the closed interval $[0,1]$. *Complementary domain* of a plane continuum X is any component of the complement of X.

The *Sierpiński triangle* [Fig. 1] is geometrically defined as follows. We take a solid equilateral triangle T_0, partition it into four congruent equilateral triangles and remove the interior of the middle triangle to obtain a continuum T_1. We proceed in the same manner with the three remaining triangles step by step to get a nested sequence $(T_n)_{n=0}^{\infty}$. The intersection $T = \bigcap T_n$ is called the Sierpiński triangle.

For our purposes a topologically equivalent definition of the Sierpiński triangle will be useful. We take a countable power $\{0,1,2\}^{\mathbb{N}}$ of a three elements discrete space with the usual Tychonoff topology and identify a point $(a_1,\dots,a_n,i,\bar{j})$ with $(a_1,\dots,a_n,j,\bar{i})$ for any $i,j,a_1,\dots,a_n \in \{0,1,2\}$ and $n \in \mathbb{N}_0$. Such a quotient is homeomorphic to the Sierpiński triangle [6], whereas the vertices of the triangle correspond to the points $(\bar{i}) = (i,i,\dots)$ for $i \leqslant 2$.

If we take two copies T and T' of the Sierpiński triangle with vertices v_0, v_1, v_2 and v_0', v_1', v_2' respectively and identify each point v_i with v_i' we get a continuum homeomorphic to the so-called *Apollonian gasket* [Fig. 2]. We give a topological characterization of the Apollonian gasket and prove that arbitrary subcontinuum with three points on the boundary is homeomorphic to the Sierpiński triangle. By doing this we solve Problem 58 from [3].

The following fact, which is due to Schönflies [1, p. 515], will be a useful tool for the consecutive characterization.

Fact 1. *Let X be a locally connected continuum in the plane. Then for every $\epsilon > 0$ there are only finitely many complementary domains of X with diameter bigger than ϵ.*

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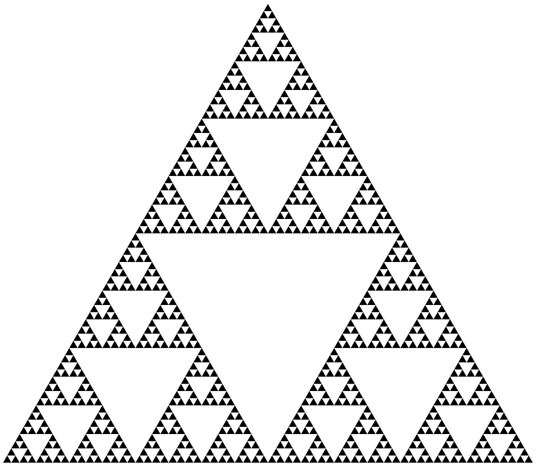

Fig. 1. The Sierpiński triangle [5].

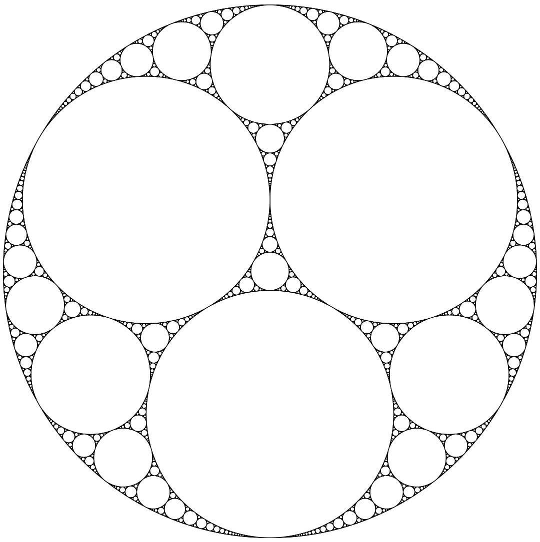

Fig. 2. The Apollonian gasket [5].

2. Main results

Definition 2. Every simple closed curve C in a continuum X will be called a *link* provided that $X \setminus C$ is connected.

This notion is especially useful when dealing with continua in the plane, because in this case every link is a boundary of a complementary domain by the Jordan curve theorem.

Theorem 3. *A continuum X is homeomorphic to the Apollonian gasket if and only if*

1. *X is planar and locally connected,*
2. *any two links in X intersect at most in a point,*
3. *there is no point in X common to three links,*
4. *X contains at least three links each pair of which intersects,*
5. *whenever there are three links each pair of which intersects, there are two other links which intersect each of the three given links.*

Proof. It is easily observed that any space homeomorphic to the Apollonian gasket satisfies all of the conditions 1–5.

Conversely suppose that X is a continuum satisfying all the five conditions. By the condition 4 there exist three distinct links C_0, C_1 and C_2 each pair of which intersects. We note that from condition 5 and condition 1 it follows that for any triple of links each par of which intersects, there are exactly two other links which intersect each of the three given links.

We denote by $A = \bigcup_{n=0}^{\infty}\{0,1,2\}^n$ a set of indices. By induction we construct a family of links $\{L(a)\colon a \in A\}$, such that

- $L(a)$ touches $L(a,b)$ whenever $a \in \{0,1,2\}^n$ and $b \in \{0,1,2\}^m$, where $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $b_1 \notin \{b_2,\dots,b_m\}$.
- $L(a)$ touches C_i whenever $i \in \{0,1,2\}$, $a \in \{0,1,2\}^n$ and $i \notin \{a_1,\dots,a_n\}$.

Let $L(\emptyset)$ be a link which touches the links C_0, C_1 and C_2. By condition 5 there are two of them so we have two possible choices. We suppose that all the links $L(a)$ for $|a| \leqslant n$ have been constructed and they satisfy the induction hypothesis. We fix $a \in A$ and $i \in \{0,1,2\}$, where $|a| = n$, and we define a link $L(a,i)$. We distinguish several cases:

- If $n = 0$, then we consider three links $L(\emptyset), C_j$ and C_k where $\{i,j,k\} = \{0,1,2\}$. These three links intersect each other. Thus by condition 5 there are two other links which touch the three given links. One of them is the link C_i. We define $L(i)$ to be the other link.
- If $n \geqslant 1$ and $|\{a_1,\dots,a_n,i\}| = 1$, we can find j and k such that $\{i,j,k\} = \{0,1,2\}$. Clearly the links $L(a)$, C_j and C_k touch each other by the induction hypothesis. Thus there are two other links touching each of these three links. One of them is $L(a_1,\dots,a_{n-1})$. We define $L(a,i)$ to be the other link.
- If $n \geqslant 1$ and $a_1 = \dots = a_n \neq i$, we can define $j = a_n$ and k such that $\{i,j,k\} = \{0,1,2\}$. The links $L(a), L(a_1,\dots,a_{n-1})$ and C_k touch each other. Thus there are two other links touching each of these three links. One of them is C_i. We define $L(a,i)$ to be the other one.
- If $|\{a_1,\dots,a_n\}| = |\{a_1,\dots,a_n,i\}| = 2$, we can find $j \in \{a_1,\dots,a_n\}$ and k such that $\{i,j,k\} = \{0,1,2\}$. Let $l \leqslant n$ be the natural number for which $a_l \neq a_{l+1} = \dots = a_n = i$. Moreover let $m \leqslant n$ be the biggest integer for which $a_m = i$. The links $L(a)$, $L(a_1,\dots,a_{l-1})$ and C_k touch each other. Thus there are two other links touching each of these three links. One of them is $L(a_1,\dots,a_{m-1})$. Let $L(a,i)$ be the other one.
- If $|\{a_1,\dots,a_n\}| = 2$ and $|\{a_1,\dots,a_n,i\}| = 3$, let us denote by $l \leqslant n$ the natural number for which $a_l \neq a_{l+1} = \dots = a_n$. Next we find the natural number $m < l$ for which $a_m \notin \{a_{m+1},\dots,a_n,i\}$. The links $L(a)$, $L(a_1,\dots,a_{l-1})$ and $L(a_1,\dots,a_{m-1})$ are three links each pair of which intersects. Thus there are two other links touching each of these three links. One of them is C_i. We define $L(a,i)$ to be the other one.
- If $|\{a_1,\dots,a_n\}| = |\{a_1,\dots,a_n,i\}| = 3$, let us denote by $l \leqslant n$ the natural number for which $a_l \neq a_{l+1} = \dots = a_n = i$. Next we find the natural number $m < l$ for which $a_m \notin \{a_{m+1},\dots,a_n,i\}$. We can find the biggest integer $p \leqslant n$ for which $a_p = i$. The links $L(a)$, $L(a_1,\dots,a_{l-1})$ and $L(a_1,\dots,a_{m-1})$ are three links each pair of which intersects. Thus there are two other links touching each of these three links. One of them is $L(a_1,\dots,a_{p-1})$. We define $L(a,i)$ to be the other one.

In each case we can easily verify that the induction hypothesis remains satisfied.

Now, we define a mapping $f : \{(a,\bar{i})\colon a \in A,\ i \leqslant 2\} \to X$ as follows:

- $f(\bar{i})$ is the only point in the intersection $C_j \cap C_k$ where $\{i,j,k\} = \{0,1,2\}$.
- If $|\{a_1,\dots,a_n,i\}| = 2$ and $a_n \neq i$ we define $f(a,\bar{i})$ to be the only point in $L(a_1,\dots,a_{n-1}) \cap C_k$ where k satisfies $\{a_n,i,k\} = \{0,1,2\}$.
- If $|\{a_1,\dots,a_n,i\}| = 3$ and $a_n \neq i$ we find the natural number $l < n$ for which $a_l \notin \{a_{l+1},\dots,a_n,i\}$ and we define $f(a,\bar{i})$ to be the only point in $L(a_1,\dots,a_{n-1}) \cap L(a_1,\dots,a_{l-1})$.

We consider the family of links $\{C_0,C_1,C_2\} \cup \{L(a)\colon a \in A\}$ and we enumerate as $\{D_m\colon m \in \mathbb{N}\}$ a family of all closures of complementary domains of these links which do not intersect X. We observe that the diameters of the sets D_m converge to zero by Fact 1. The assumptions of Fact 1 are satisfied because of the condition 1.

In order to show that the mapping f is uniformly continuous it suffices to prove that for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that the components of $\mathbb{R}^2 \setminus \bigcup\{D_m\colon m < n\}$ are of diameter less than ε. Suppose that this is not true. Hence there is an $\varepsilon > 0$ and there are complementary domains G_n of $\mathbb{R}^2 \setminus \bigcup\{D_m\colon m < n\}$ whose diameters are at least ε and such that $G_{n+1} \subseteq G_n$. For every $m \in \mathbb{N}_0$ there are three mutually distinct indices p_m, q_m and r_m in $\mathbb{N}$ such that G_m is the bounded complementary domain of $\mathbb{R}^2 \setminus (D_{p_m} \cup D_{q_m} \cup D_{r_m})$.

By eventual restriction to a subsequence of (G_m) and possible permutation of p_m, q_m and r_m it suffices to consider the following three cases only.

- The sequence (p_m) goes to infinity and (q_m) and (r_m) are constant. We denote by x the only point in $D_{q_0} \cap D_{r_0}$. As the diameters of (D_{p_m}) converge to zero we get that (D_{p_m}) converges to the point x. Hence the sets G_m go to x and thus the diameters of G_m converge to zero.
- The sequences (p_m) and (q_m) tend to infinity and (r_n) is constant. There is a point $x \in X$ which is a limit point to the sequence $(D_{p_n} \cup D_{q_n})$. Similarly as in the first case we derive that the sequence G_m converge to the point x, hence its diameters tend to zero.
- All the sequences $(p_m), (q_m)$ and (r_m) converge to infinity. Then we get that the diameter of G_m is less than or equal to the diameter of $D_{p_m} \cup D_{q_m} \cup D_{r_m}$ which converges to zero.

In all cases we obtain a contradiction.

We denote by $g:\{0,1,2\}^{\mathbb{N}} \to X$ the only continuous extension of the mapping f. It follows using the condition 2 that $g(a_1,\dots,a_n,i,\bar{j}) = g(a_1,\dots,a_n,j,\bar{i})$ and that these are the only possibilities when $g(x)=g(y)$ for $x \neq y$, because of the condition 3. Thus the image of $\{0,1,2\}^{\mathbb{N}}$ under g is homeomorphic to the Sierpiński triangle.

Now we recall that there were two possibilities $L(\emptyset)$ and $L'(\emptyset)$, how to choose the first link in the inductive process. Thus we may obtain by the same proof another family of links $\{L'(a)\colon a \in A\}$ and corresponding mapping f' and its continuous extension g'.

By a similar reasoning as in the proof that f is uniformly continuous we can show, that the union of images of the mappings f and f' is dense in X. And thus the union of the images of g and g' covers the whole space X. The intersection of the images of the mappings g and g' consists of three points. These are namely the points contained in exactly two links from C_0, C_1 and C_2. Thus we obtain that X is homeomorphic to the quotient of the direct sum of two copies of the Sierpiński triangle, where the corresponding vertices are identified. Thus X is homeomorphic to the Apollonian gasket. $\square$

Observation 4. Since the choice of the three links C_0, C_1 and C_2 at the beginning of the preceding proof was random, we conclude that for any triple of distinct, mutually intersecting links C_0', C_1' and C_2' there is a homeomorphism of the Apollonian gasket onto itself, which sends C_i onto C_i' for any $i \leqslant 2$.

Lemma 5. *Let X be a locally connected continuum and K be a non-degenerate subcontinuum of X with finite boundary. Then every point from the boundary of K is a local cut-point of X.*

Proof. Let x be an arbitrary point from the boundary of K. Since the boundary of K is finite and X is locally connected, there is an open connected neighborhood U of x whose intersection with the boundary of K contains only the point x. We get that the set $U \setminus \{x\}$ is a disjoint union of an open set $K \cap U \setminus \{x\}$ and an open set $U \setminus K$. Both these sets are non-empty since x is an element of the boundary of K. Thus x is a cut-point of U and consequently it is a local cut-point of X. $\square$

Theorem 6. *Any subcontinuum of the Apollonian gasket whose boundary consists of exactly three points is homeomorphic to the Sierpiński triangle.*

Proof. Suppose that X is a subcontinuum of the Apollonian gasket with precisely three points v_0, v_1, v_2 on the boundary. By Lemma 5 we know that the points v_0, v_1, v_2 are local cut-points, which are those points in the Apollonian gasket, where two distinct links intersects. For any pair of distinct indices $i,j \in \{0,1,2\}$ there is a complementary domain D_k, where $\{i,j,k\} = \{0,1,2\}$ and the points v_i and v_j belong to the boundary of D_k. The boundaries C_0, C_1 and C_2 of D_0, D_1 and D_2 are pairwise distinct links and $\{v_i\} = C_j \cap C_k$ for $\{i,j,k\} = \{0,1,2\}$. By Observation 4 we conclude that X is homeomorphic to the Sierpiński triangle. $\square$

Corollary 7. *Let X be a continuum and let v_0, v_1, v_2 be three points in X. Define a space Y as a sum of X and a disjoint copy X' of X, where every point v_i is identified with v_i' which is a corresponding point in X'. If the space Y is homeomorphic to the Apollonian gasket, then X is homeomorphic to the Sierpiński triangle.*

Proof. Clearly X is a subcontinuum of the Apollonian gasket Y and its boundary consists of three points. Thus by Theorem 6 it follows that X is homeomorphic to the Sierpiński triangle. $\square$

3. Final remarks

Let us define a continuum T, that will be called a *modified triangle*, in the following way. We take an equilateral triangle and exclude the interior of a regular hexagon whose three edges are formed by the middle thirds of edges of the triangle. This can be inductively done in every remaining smaller triangle. What remains is the modified triangle [Fig. 3]. A *modified gasket* is a sum of T and its copy T' where the corresponding pairs of vertices of the triangles are joined with an arc. This can be pictured as in Fig. 4.

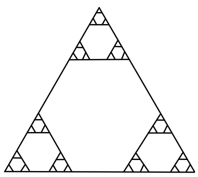

Fig. 3. The modified triangle.

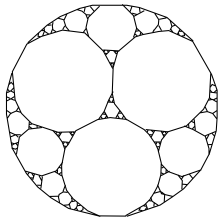

Fig. 4. The modified gasket.

Table 1
A comparison of some properties of selected continua.

	Number of topologically distinct embeddings into the plane	Contains a dense set of topologically equivalent points
Sierpiński triangle	infinite	no [6]
Sierpiński carpet	one	yes [2]
Apollonian gasket	one	yes

A simple modification of the second condition in the characterization of the Apollonian gasket from Theorem 3 gives rise to a characterization of the modified gasket. Now, there is even no need to include a parallel to the third condition from Theorem 3.

Theorem 8. *A continuum X is homeomorphic to the modified gasket if and only if*

1. *X is planar and locally connected,*
2. *any two links in X are either disjoint or their intersection is an arc,*
3. *X contains at least three links each pair of which intersects,*
4. *whenever there are three links each pair of which intersects, there are two other links which intersect each of the three given links.*

The reader may be confused by that we didn't give any 'direct' characterization of the Sierpiński triangle. This is partially explained by Table 1 where a comparison with two other more or less related continua is given. There are some crucial differences between the Sierpiński triangle on one side and the two other continua which possess nice direct characterizations on the other side. We have shown a direct topological characterization of the Apollonian gasket. The *Sierpiński carpet* [1, p. 275], which arise from a solid square by partitioning it into 9 congruent squares, eliminating the central one and repeating this process inductively in all 8 remaining squares, is characterized as a one-dimensional locally connected planar continuum with no local cut-points [4]. We believe that there is no nice internal characterization of the Sierpiński triangle.

The Sierpiński triangle is generalized in [6] in the following way. We take any $n \in \mathbb{N}$ and we consider a space T_n obtained as a quotient of $\{0, 1, \ldots, n-1\}^{\mathbb{N}}$ where every sequence $(a_1, a_2, \ldots, a_m, i, \bar{j})$ is identified with $(a_1, a_2, \ldots, a_m, j, \bar{i})$ for $i, j, a_1, \ldots, a_m \in \{0, \ldots, n-1\}$ and $m \in \mathbb{N}_0$. We gave in Theorem 6 a topological characterization of the Sierpiński triangle which is homeomorphic to T_3. Thus there is another natural problem related.

Problem 9. *Give a topological characterization of the space T_4.*

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