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Some Physical Properties of Self-Affine Rough Surfaces.

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Abstract. – We study some examples where the self-affine nature of surfaces determines the scaling of transport properties, beyond the simple geometrical characterization. The first example deals with the permeability of two identical surfaces which have been translated with respect to each other along their mean orientation. The second example is the force displacement characteristic of two elastic solids limited by independent self-affine surfaces which are pressed against each other.

Introduction. – In recent years, there has been a considerable interest in the geometry of systems appearing to be scale invariant, or fractal [1]. More recently, *self-affinity*, *i.e.* a more general scaling transformation which takes anisotropy into account, has been found to appear naturally in quite a number of different areas. Examples of this may be found in growth models—such as the boundary of Eden clusters, or ballistic-deposition models recently reviewed in ref. [2]—landscape and erosion surfaces [3], or fracture surfaces [4-7]. It is indeed important to be able to identify such a geometrical property whenever it exists, but often it is not clear what *physical consequences* this geometrical property may imply. It is the aim of this letter to identify a few of such physical properties whose scaling behaviour can be simply attributed to the underlying self-affine geometry.

A self-affine object is invariant under an affine transformation: $x_i \to \lambda_i x_i$, for $i = 1, ..., d$. This invariance is in general to be considered only in a *statistical* sense. For this symmetry to be meaningful, a reasonable range of λ values should fulfil this condition. Requiring that these transformations can be combined implies a group structure which results in each λ_i to be an homogeneous function of one of them, say λ_1. The homogeneity exponent is called ζ_i:

$$\lambda_i = \lambda_1^{\zeta_i}. \tag{1}$$

The set of homogeneity indices $\{\zeta_i\}$ characterizes the scaling properties of the self-affine object. In the following, we will only consider the case of surfaces with a mean plane parallel to $(x_1, \ldots, x_{d-1})$ along which the surfaces are isotropic. The only non-trivial exponent is thus relative to the scaling in the x_d direction ζ_d—hereafter simply referred to as ζ. In the rest of this letter, we will keep the discussion on these well-defined notions, and avoid the use of any «fractal-dimension» concept which can assume different values depending on the precise definition or measurement method chosen. This term is also somewhat misleading in the sense that it unduely encourages the use of tools and results which were developed specifically for self-similar objects (*i.e.* $\zeta_i = 1$ for all i).

The examples we consider in this letter concern fracture surfaces. These have been observed to exhibit self-affine geometries under very general conditions [4-7]. Furthermore, the roughness exponent has been found to be surprisingly insensitive to the materials involved, and how they were fractured. The typical value found is $\zeta \approx 0.85$ [6, 7]. We first study the permeability of a crack which has been subjected to a displacement parallel to its mean surface. As a second example, we consider a semi-infinite elastic medium limited by a self-affine surface which is pressed onto another independent self-affine surface. We finally consider the transport properties across the interface in the previous situation.

Consider a single rough crack in a solid. We assume that the two opposite parts of the solid are translated with respect to each other by an amount d parallel to the mean crack plane (with no rotation). We moreover suppose that the solid is undeformable, or that the applied normal stress pressing the two sides of the crack together is low, so that most of the deformation takes place in the relative displacement of the two rigid blocks. In order to allow a displacement d without deformation of (nor overlap between) the two blocks, a normal displacement $h(d)$ is necessary. If we introduce a parametrization of the surface $z(x)$ (for $x = (x_1, \ldots, x_{d-1})$), the normal displacement can easily be computed to be

$$h(d) = \max_x (z(x) - z(x+d)). \tag{2}$$

The self-affinity of the surface imposes that $h(d) \propto |d|^\zeta$. The local opening of the crack will be an essential feature which controls the flow properties discussed below. The opening $a(x) = z(x+d) + h(d) - z(x)$ can be shown to display a self-affine structure only over a limited range of length scale, which depends on d. More precisely, the pair correlation function can be cast into the scaling form

$$\langle (a(x) - a(x'))^2 \rangle = |x - x'|^{2\zeta} \varphi(|x - x'| h(d)^{-1/\zeta}), \tag{3}$$

where $\varphi(x) \sim \text{const}$ for $x \ll 1$ and $\varphi(x) \sim x^{-2\zeta}$ for $x \gg 1$. Therefore, the statistical distribution of local aperture can also be cast in a scaling form where the reduced variable $a/h(d)$ allows to capture the d dependence.

Let us now consider the laminar flow of a Newtonian fluid in the crack, in a direction parallel to the mean crack plane. Figure 1 sketches the two-dimensional version of the geometry we consider below. In this case, the flow has to take place perpendicularly to the figure plane. However, the argument we develop below applies also to the general situation where the surfaces are *not* invariant along the flow direction.

In most cases of physical interest, the roughness exponent ζ is smaller than 1. In such a case, the *relative roughness* (*i.e.* the roughness divided by the system size) tends to zero as

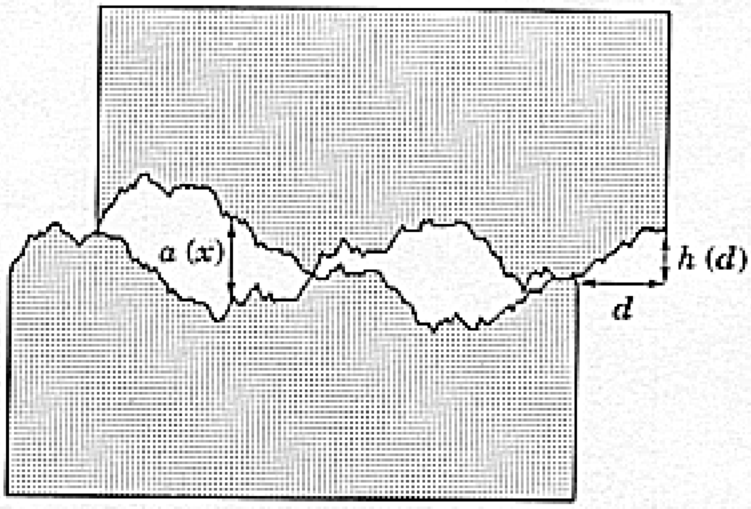

Fig. 1. – The two opposite surfaces of a crack are displaced parallel to the mean crack plane by some amount d. The material is considered to be rigid and undeformable. As a result, the two sides have to be displaced perpendicularly by $h(d)$ so as to avoid overlap. We study the flow, normal to the figure, in the space which is comprised between the two surfaces, $a(x)$ and, more specifically, the scaling of the permeability with $|d|$.

the system size increases to infinity. This observation implies that fracture planes are asymptotically flat. Such a property allows us to simplify the estimate of the permeability scaling. The asymptotic behaviour (for large system size) can be obtained with arbitrarily small prefactors in the roughness, and thus the roughness can be treated as a perturbation on the simple Poiseuille flow between parallel plates.

This allows to derive the scaling of the permeability as a function of the displacement d. Locally, the hydraulic conductance scales as the cube of the aperture, $K(x) \propto a(x)^3$. When the two opposing faces of the crack are translated by a quantity d parallel to the mean fracture plane, the distribution of local apertures has the same form for all d, apart from a scaling factor which depends on $|d|$ as mentioned above. Since the upper cut-off in the aperture distribution is proportional to $|d|^\zeta$, the *entire* aperture distribution scales as $|d|^\zeta$. Thus, the scaling of the local conductance distribution is simply $|d|^{3\zeta}$. As a consequence, we conclude that the global hydraulic conductance K for a unit length along the flow and perpendicular to it scales as

$$K \propto |d|^{3\zeta} . \tag{4}$$

The permeability is deduced from the hydraulic conductance by taking into account the cross-section available for the flow. Thus the hydraulic conductance should be divided by the mean aperture to give the permeability, thus it scales as

$$\kappa \propto |d|^{2\zeta} . \tag{5}$$

However, the latter quantity requires the knowledge of the mean aperture, which may be difficult to estimate experimentally for each displacement. Thus, the roughness exponent determines the scaling of the permeability as a function of the displacement. With a roughness exponent $\zeta = 0.85$, we find that $K \propto |d|^{2.55}$.

There has been a number of theoretical and experimental investigations of the permeability of cracks [8,9]. However, in most cases, the key problem addressed was the evolution of the permeability as a function of the aperture of a crack, under a varying normal pressure. As the pressure increases, the closing of the crack leads to an interesting percolation problem where, however, the self-affinity of the surface leads to long-range

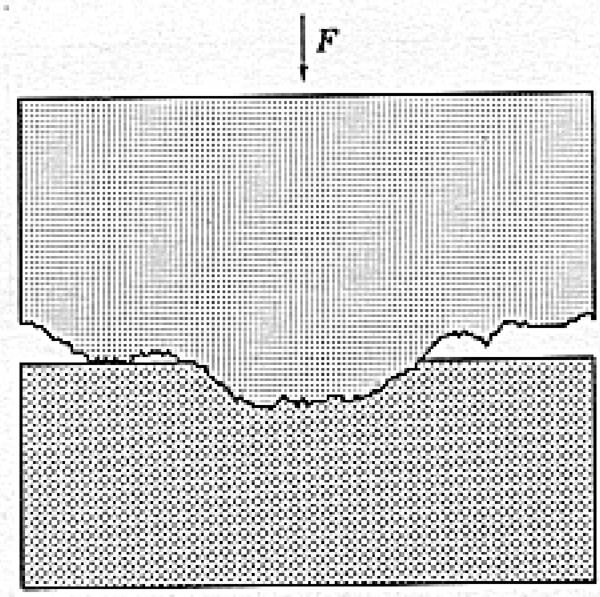

Fig. 2. – Schematized is the elastic contact between a flat plane and the self-affine boundary of a rigid punch. As the normal force F is varied, the interpenetration between the two solids δ follows a non-linear relation with respect to F, whose form depends on the roughness exponent ζ.

correlations which are expected to change the value of the critical exponents compared to the usual accepted values [10]. This point—which deserves a detailed study—is however far from our initial problem. This has not been studied so far to our knowledge.

Let us now consider a semi-infinite elastic solid limited by a self-affine surface with an exponent ζ_1. This domain is pressed against another elastic solid limited by an *independent* self-affine surface with an exponent ζ_2, whose mean orientation is parallel to the first surface. As shown in fig. 2, a normal force F is applied, so as to close the space in between the two surfaces. As F is increased, the area of contact between the two solids will increase, and thus the force displacement is non-linear, as in the classic example of two spheres in contact solved by Hertz in the last century [11]. Correlations in the height distribution will appear to be determinant as shown below.

First, it is important to note that, as for the Hertz contact, the two surfaces only appear through the distance which separates them. In our case, this separation will also be self-affine, with an exponent $\zeta = \max(\zeta_1, \zeta_2)$. Therefore, this problem can be reduced to the indentation of an elastic half-space by a self-affine rough surface as shown in fig. 2. Similarly, the elastic properties of the two solids only come into play through the harmonic average of plane strain elastic moduli [11]. We will derive the force displacement scaling by using the self-affinity of the separation, and the combination of two simple transformations.

The first group of transformations consists of uniform dilations: $x_i \rightarrow \lambda x_i$ $(i = 1, ..., d-1)$ and $z \rightarrow \lambda z$ along and perpendicularly to the mean surface. Under such transformations, both the strain ε and the stress σ remain unchanged, the former because of the geometric transformation and the latter due to the linearity of the elastic behaviour. The contact area will scale as $S \rightarrow \lambda^{d-1} S$. Therefore, the total force scales as the contact area times the stress: $F \rightarrow \lambda^{d-1} F$.

The second group of transformations consists of affinities along the z-axis, $z \rightarrow \mu z$, while *the other directions remain unchanged*. Under those transformations, the fundamental property is that the contact area is now unchanged, whereas the penetration δ is changed as $\delta \rightarrow \mu\delta$. This means that the loading varies linearly with μ with a constant area of contact. As a consequence of the linearity of the elastic behaviour, the normal stress under the contact will scale as $\sigma_N \rightarrow \mu\sigma_N$. The total force scales as the normal stress times the area, $F \rightarrow \mu F$. Let us emphasize that the non-linearity of the contact law comes from the variation of the contact area with the applied force. In the latter affine transformations, this variation is cancelled.

None of those two transformations, dilation and z-affinity, preserve the statistical invariance of the separation. This can, however, be achieved by combining these two groups of transformations successively with $\mu = \lambda^{\zeta - 1}$, so that $x_i \to \lambda x_i$ and $z \to \lambda^{\zeta} z$. In this case, the force F, the interpenetration δ and the area of contact S will be rescaled as

$$\begin{cases} \delta \to \lambda^{\zeta} \delta , \\ S \to \lambda^{d-1} S , \\ F \to \lambda^{d+\zeta-2} F . \end{cases} \tag{6}$$

Searching for a combination of those quantities which are invariant for all λ we find the two scaling relations:

$$\begin{cases} F \propto \delta^{(\zeta+d-2)/\zeta} , \\ S \propto \delta^{(d-1)/\zeta} , \\ S \propto F^{(d-1)/(d+\zeta-2)} . \end{cases} \tag{7}$$

Using this simple scaling approach, we can recover the classic Hertz law for two spheres in contact [11]. In this case, the ζ exponent is 2 ([1]), and the space dimension $d = 3$, therefore $F \propto \delta^{3/2}$. The value $\zeta = 2$ comes from the fact that at the contact point the surfaces can be approximated by parabolas which are invariant under the affinity $x_i \to \lambda x_i$ for $i = 1, ..., d-1$ and $z \to \lambda^2 z$, with a fixed radius of curvature.

With a fracture roughness exponent 0.85, we find that $F \propto \delta^{2.2}$. It is interesting to confront this expectation with experimental results which have been collected in the past on this problem. A power law relation between the force and the displacement is indeed very commonly encountered in compression tests on fracture surfaces [12]. The typical exponent which is measured is in the range $2 \div 3$, in good agreement with our expectation considering the usual uncertainty associated with these measurements. To our knowledge, this provides the first theoretical explanation for this long-standing empirical observation.

Note that for the case of real fractures, the most common situation to be encountered is when the two facing self-affine surfaces are not independent, but rather opposite faces of the same crack, which may be translated with respect to each other as in fig. 1, by an amount d along the mean fracture plane. In this case, the early stage of the compression should display the scaling law obtained in eqs. (5). Above a crossover interpenetration, $\delta = \delta^* \propto |d|^{\zeta}$, the identity of the two surfaces becomes apparent, and the force is expected to increase very abruptly (faster than any power law).

It is amusing to note that in the two-dimensional case the force displacement relation does not depend on the roughness exponent any longer. An exact computation can be performed for two parallel cylinders ($\zeta = 2$) which shows that an additional logarithmic contribution is to be expected [11]. This logarithmic correction is obviously out of reach of the present dimensional analysis.

In the latter example, one could also study transport from one edge of the crack to the opposite one, across the surface. In a number of cases, the void in between the two

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([1]) For the Hertz problem, the separation between the two contacting surfaces is approximated by the kissing parabola $z = \sum a_{ij} x_i x_j$ which can be fitted to the formalism of the self-affine surfaces using $\zeta = 2$ so as to preserve the curvature.

boundaries is insulating, whereas the solid is conducting. One could think of the example of heat transport. In such a case, using again the asymptotic flat character of the surface, one is led to the conclusion that the transport coefficient between the two solids is simply proportional to the area of contact. We have seen in eq. (5) that the latter varies as a power law of the normal force, with an exponent $(d-1)/(d+\zeta-2)\approx 2.35$ in three dimensions for real fractures. Thus, the conduction across the crack appears to display again a non-trivial dependence on the normal pressure which results from the self-affine nature of the crack geometry.

Other examples of physical properties which exhibit a non-trivial scaling due to the self-affine character of the surface should be investigated, both for a better understanding of the considered physical phenomena themselves, and also for an indirect way of detecting some geometrical features when they are not directly accessible through a direct measurement.

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