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Multifractality of the current distribution in directed percolation

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Abstract. We investigate numerically the distribution of currents in a two-dimensional diode network at the directed percolation threshold. We obtain the multifractal spectrum of this distribution as well as other scaling properties such as the fractal dimension of the backbone, and the scaling of the conductivity. The analysis of the multifractal spectrum provides information about the singly connected bonds in the lattice.

1. Introduction

The conductivity properties of diode networks close to the directed percolation threshold have been studied theoretically [1], numerically [2] as well as experimentally [3] in a series of studies by Redner *et al* between 1981 and 1983. Since then, much attention has been paid to the exhaustive characterization of the transport properties in disordered systems, such as random resistor networks close to the percolation threshold. In particular, the powerful concept of the multifractality [4] of the local current distribution has been evidenced and used to provide a detailed description of all critical properties related to transport [5, 6]. A few generalizations and applications of these concepts were recently brought forward, but in most cases this has been in relation to linear transport properties.

The purpose of the present study is to investigate the critical behaviour of the conductivity as well as the scaling of the current distribution in the context of directed percolation, i.e. in a random diode network at threshold. Despites the nonlinearity of the problem, most features of non-directed percolation can be recovered, as we will show later.

After a short description of the problem and the way we handled it numerically, we will report the results obtained on the geometrical properties of the network (section 2) and on the transport properties (section 3).

2. Geometrical properties

Let us consider a square lattice whose bonds are simple diodes (with a constant conductance of one for a positive voltage, and perfectly insulating for negative voltage).

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The overall shape of the lattice is square (size $L \times L$), and the principal axes of the lattice are oriented at 45° with respect to the borders. The top and bottom borders are considered as equipotential electrodes between which a voltage drop is set, whereas periodic boundary conditions are implemented on the lateral borders. All diodes are oriented in such a way that the current may flow from bottom to top, but not backward. The network is then diluted at random, with a probability of presence for each bond equal to p. In all the simulations reported here, the probability p was set to the directed bond percolation threshold $p = p_c \approx 0.6447$ [7].

Let us first introduce the geometric features of interest for the determination of the transport properties. Two sets of sites are important, and figure 1 illustrates their construction. First, the set of sites (referred to as $\mathcal{A}$) which are connected to the bottom border via a directed path. Second, the set of sites ($\mathcal{B}$) from which it is possible to reach the top border. These two sets can be constructed in the same way by reversing the orientation of the diodes, and exchanging the top and bottom borders. There are many ways in which to generate these sets numerically. We use the procedure described in [8].

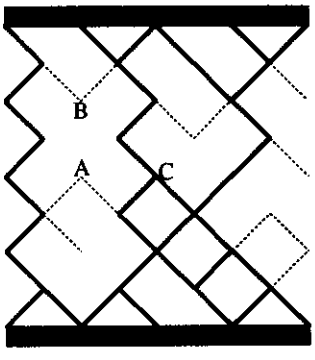

Figure 1. Schematic drawing of a lattice. The bonds in bold or dotted lines are present bonds which can only carry a current flowing upward. The sites such as A and C can be reached from the bottom bus bar using a directed path, and they are part of a set called $\mathcal{A}$. Site B cannot be reached without following a backward step, and thus it does not belong to $\mathcal{A}$. Starting from sites B and C, one can reach the top bus bar through directed paths, and thus these sites belong to set $\mathcal{B}$. Site A does not belong to this set. The intersection of sets $\mathcal{A}$ and $\mathcal{B}$ forms the geometrical backbone (shown in bold lines). Indeed, a non-zero current may flow at site C but not at A or B because of the directedness constraint.

We first checked the well known scaling of the density of these sets as a function of the system size. The probability, $P_{\mathcal{A}}$, of belonging to set $\mathcal{A}$ in an infinite lattice approaches zero as p approaches p_c as $P_{\mathcal{A}} \propto (p - p_c)^{\beta}$, where in two dimensions, $\beta \approx 0.277$. Away from threshold, two correlation lengths can be defined, either parallel, $\xi_{\|}$, or perpendicular, $\xi_{\perp}$, to the preferred direction. These two length scales diverge at threshold with different scaling exponents $\nu_{\|} \approx 1.7334$ and $\nu_{\perp} \approx 1.0972$ [7]. At threshold, we can use the usual finite-size arguments to obtain an expression for the probability $P_{\mathcal{A}}$.

For a rectangular lattice of size $L_{\|} \times L_{\perp}$ at threshold, effective correlation lengths can be defined such that $\xi_{\|} \leqslant L_{\|}$ and $\xi_{\perp} \leqslant L_{\perp}$ and $\xi_{\|}^{1/\nu_{\|}} = \xi_{\perp}^{1/\nu_{\perp}}$. The largest correlation lengths fulfilling these conditions have to be chosen. In the case of a

square geometry, $L_{\|} = L_{\perp} = L$, $\xi_{\|}$ will be equal to L. In the perpendicular direction, for length scales larger than $\xi_{\perp}$ and smaller than L, the system appears to be homogeneous. These classical results allow the probability to belong to set $\mathcal{A}$ to be expressed as a function of L for a square lattice: $P_{\mathcal{A}} \propto L^{-\beta/\nu_{\|}}$; and thus the mass of the set $\mathcal{A}$ scales as $L^{2-\beta/\nu_{\|}}$. Numerically, the latter scaling exponent amounts to 1.84.

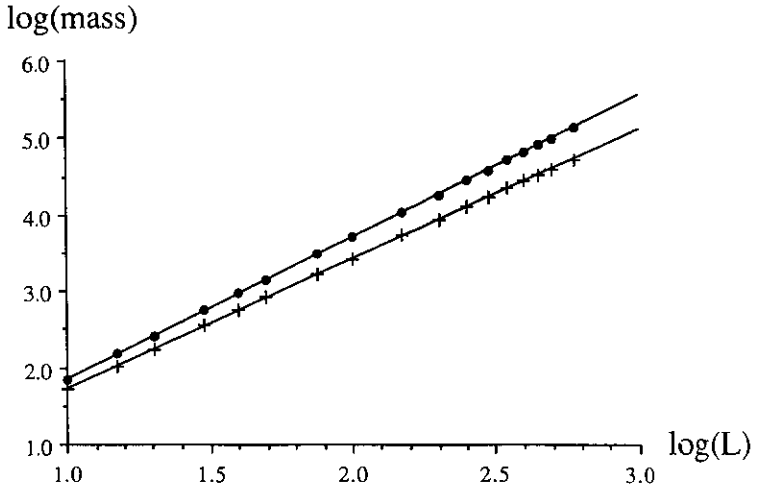

Figure 2. Log–log plot of the mass of the set $\mathcal{A}$ (•) and of the geometrical backbone (+) as a function of the system size L. The straight lines through the data are best fits, with respective slopes 1.86 and 1.70.

Figure 2 shows the numerical results obtained for sizes L ranging from 10 to 600, and a number of realizations varying from 10 000 to 5000 for the two extreme sizes. The estimate of the 'fractal dimension' of the set $\mathcal{A}$ is approximately 1.86, in agreement with the expected value. We use here the expression 'fractal dimension' because of the power-law relationship between the mass of the set and the system size, although due to the anisotropic character of the problem, the set is not self-similar, but rather self-affine as long as the system size does not extend over the correlation lengths. We will come back to some consequences of the anisotropy of the network in the discussion of the results relative to the electrical properties of the networks.

An interesting set can be defined in relation to the conductivity properties: the 'geometrical backbone'. This set is defined as the intersection of the two sets $\mathcal{A}$ and $\mathcal{B}$. Indeed a site can only be part of the backbone (current-carrying part of the lattice) if it can receive a current from the bottom electrode and if it can forward it to the top electrode. An example of such a structure is shown on figure 3 for a lattice of size 90. An interesting property, first noted by Arora *et al* [9], is that for any site in the lattice, the probability of belonging to $\mathcal{A}$ or $\mathcal{B}$ is independent. Indeed these two probabilities depend on the presence or absence of bonds in two disconnected sets (i.e. upper and lower cones whose summits are the point of interest). Thus the probability to belong to the geometrical backbone can be simply expressed as the product of the probabilities to belong to one set or the other. As a simple consequence the fractal co-dimension of the backbone is twice that of the set $\mathcal{A}$. The mass of the backbone simply scales as $L^{2-2\beta/\nu_{\|}}$, thus with dimension 1.68. We have reported in figure 2

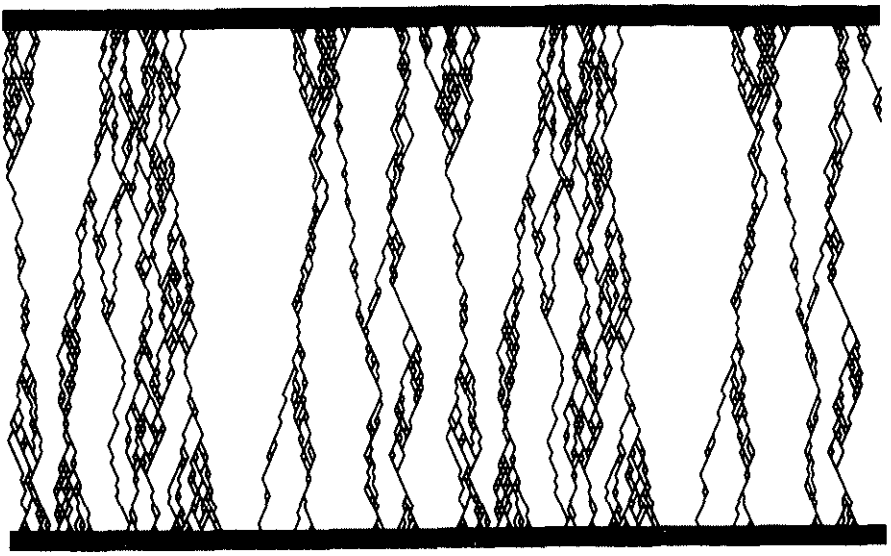

Figure 3. An example of a geometrical backbone obtained for a lattice of size 90. The electrical backbone is a subpart of this structure, that can only be obtained after the determination of the current distribution.

the data relative to the backbone, and a linear regression through all points provides the estimate 1.70 in agreement with the expected result.

3. Electrical properties

3.1. Numerical technique

In order to investigate the transport properties of the network, we first need to solve the current distribution within the network. This solution turns out to be a very time consuming part of the simulations, and in the rest of the study, we had to reduce considerably the range of sizes considered ($10 \leqslant L \leqslant 50$), as well as the number of realizations (varying from 12000 for $L = 10$ to 100 for $L = 50$).

The first step in the simulation is to extract the geometric backbone which necessarily contains the actual backbone. Then we solve the current distribution by assuming a constant conductance in each bond (as if they were resistors) and a unit voltage drop between the bottom and top electrodes. We use a conjugate gradient algorithm to solve this elementary linear problem, with an accuracy of 10^{-10}. Then for all bonds we check that the current is indeed flowing in the allowed direction. For lattice sizes above about 10, in most cases, some bonds are found to carry a backward current. In this case, these bonds are then considered to be perfect insulators, and the currents are recomputed according to this new status. With this updated current distribution, we again checked throughout the lattice that all conducting bonds carry a positive current, and that all present bonds that were set to an insulating state previously are subjected to a negative voltage drop. When these conditions are violated, the status of the bond is corrected and the current recomputed. Those two last steps are repeated until convergence.

Although the bonds which differ between the geometric and electric backbones, are not numerous, their number increases rapidly with system size. Thus the procedure described earlier becomes more and more time consuming. A quick estimate of the increase in the computation time with L reveals that the dependence is exponential, although each basic step is algebraic in time.

However, it can be shown that the solution of the potential distribution on the structure is unique, and our procedure always converges toward a distribution which fulfils the elementary diode characteristic of each bond. Thus the distribution obtained is the solution for which we were looking.

3.2. Results

Two types of data were collected through the simulations. First various moments of the current distribution were computed (order ranging from 0 to 4 by steps of 0.5) and, second, the total histogram of the current distribution was recorded.

Some moments of the current distribution have a simple physical interpretation. The zeroth-order moment is simply the number of bonds which belong to the electrical backbone. Although the number of bonds which are removed from the geomerical backbone increases significantly with system size, it still represents a small fraction of the total for the maximum size investigated ($L = 50$). A linear regression through the log–log plot of the data provided an estimate of the fractal dimension of the actual backbone: 1.67 which seems to be strictly smaller than that of the geometrical backbone since, due to finite-size corrections, the effective exponent of the geometrical backbone over the same set of sizes was somewhat larger than the previously reported value. However, it is difficult to be definitive on this conclusion. The first moment is trivial to obtain directly: the sum of currents flowing accross any line parallel to the electrodes is equal to the total current, I. Thus the first moment is simply LI. Since a unit voltage drop is set across the network, I is equal to the conductance of the network, G. Thus the first moment is equal to $M_1 = LG$. The second moment is the total energy dissipated in the lattice (up to a factor $\frac{1}{2}$). It is thus equal to $M_2 = \frac{1}{2}G$. From the scaling of the first or second moment, we obtain an estimate of the scaling of the conductance: $G \propto L^{-t/\nu_\parallel}$, with $t/\nu_\parallel \approx 0.38$. This result is consistent with the results of Redner *et al* [13] in which $t/\nu_\parallel \approx 0.34$, and Arora *et al* [9] ($t/\nu_\parallel \approx 0.42$). The fourth moment of the current distribution can be related to the critical behaviour of the resistance noise in the system [10]. Our data indicate that the scaling exponent of the fourth moment will be $M_4 \propto L^{-b/\nu_\parallel}$ with $b/\nu_\parallel \approx 1.05$.

As mentioned in the introduction, one of the most important features evidenced in non-directed percolation is the multifractal character of the current distribution in a random resistor network. We refer the reader to [4, 6] for a detailed presentation of multifractality. Let us simply mention the basic result: The distribution of the logarithm of the currents, $\mathcal{N}(j, L)$, shows a lattice-size dependence which can be accounted for by using the reduced variables: $f = \log(\mathcal{N}(j, L))/\log(L)$ and $\alpha = -\log(j)/\log(L)$. The function $f(\alpha)$, called the multifractal spectrum, turns out to be *size independent*, and thus to account for all scaling properties of the current. f can be interpreted physically as being the fractal dimension of the set of bonds whose current scales with the system size as $L^{-\alpha}$. In addition, the Legendre transform of the multifractal spectrum gives the scaling exponents of the moments as a function of the moment order.

Figure 4 shows three histograms of the current distribution for three sizes $L = 10$, 20 and 40, using the rescaled variables f and α. We see that the smaller system size differs, but that the other two spectra are indistinguishable. The quality of the data collapse shows that these distributions are indeed multifractal.

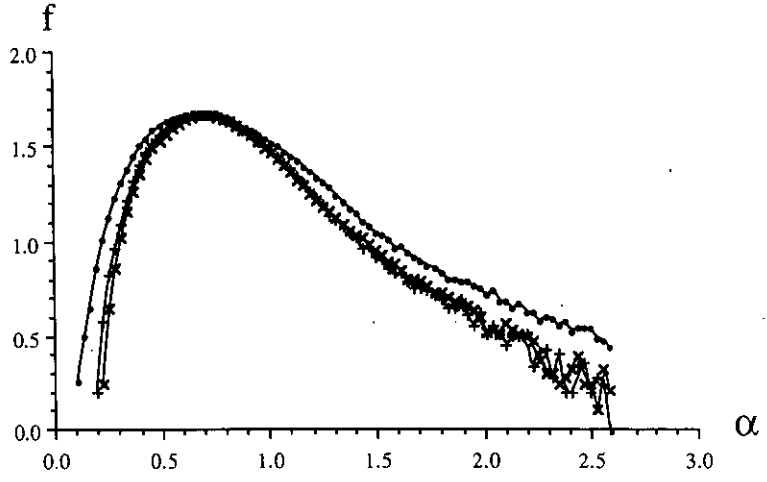

Figure 4. Rescaled histograms of the current distribution corresponding to three lattice sizes: $L = 10$ (•); $L = 20$ (+); and $L = 40$ (x) using the rescaled variables f and α defined in the text. These data points fall onto a unique curve called the multifractal spectrum.

The spectrum $f(\alpha)$ contains all the necessary information for computing the scaling of all moments in the network. In particular, it is tempting to use this function to extract some information relative to the singly connected bonds within the lattice. The singly connected bonds are the bottlenecks which concentrate all the current in a blob whose dimensions are those of the correlation lengths. Due to the anisotropy of the directed percolation problem, in a lattice of size $L \times L$, the notion of singly connected bonds is ill defined; the effective transverse correlation length is of the order of $L^{\nu_\perp/\nu_\parallel}$ and thus is much smaller that L. Thus the current flowing through these bonds should be of the order of $L^{-\alpha_{\min}} \propto L^{\nu_\perp/\nu_\parallel - 1}$. However due to the fact that the conductance fluctuates, the actual value of $\alpha_{\min}$ also fluctuates. Consequently, the end point of the spectrum, which is expected to lie at a point of coordinates $(\alpha_{\min}, f_{\min})$ asymptotically (i.e. as L goes to infinity), will become a continuous curve which can be related to the statistical distribution of the conductivities of the lattice. It is thus difficult to extract precise information which is free of the finite-size correction. Indeed we can see from figure 4 that the minimum value of α is smaller than expected ($1 - \nu_\perp/\nu_\parallel \approx 0.37$). However, we may roughly estimate the fractal dimension of the singly connected bonds, by considering the value of f achieved for the theoretical $\alpha_{\min}$. We obtain the estimate $f \approx 1.1$, which should be considered to be subjected to large uncertainty.

Let us note here a particularity of the directed percolation problem: Due to the anisotropic character of the correlation lengths, one alternative choice to the overall square geometry of the lattice is to consider strips of length $L_\parallel$ and width $L_\perp$, with a power-law relation between these length such that $L_\parallel \propto L_\perp^{\nu_\parallel/\nu_\perp}$. Such a

geometry was considered in previous numerical and experimental studies by Redner [2, 3]. With this geometry, the multifractal spectrum of the current distribution will be different, although it is a simple matter to take this factor into account. With the rectangular geometry, let us define f and α using the logarithm of the length of the lattice as a scale factor. The physical picture to keep in mind in order to connect the spectra in both geometries, is to imagine the square lattice to be a series of strips of size $L_{\|} = L$ and $L_{\perp} = L^{\nu_{\perp}/\nu_{\|}}$. The number of strips is $L/L_{\perp} \propto L^{\varphi}$ with $\varphi = 1 - \nu_{\perp}/\nu_{\|}$. Each of these strips can be considered as independent elements which are in parallel between the electrodes. Since we imposed a unit voltage drop accross the square lattice, each strip is subjected to the same boundary condition, and therefore the variable α does not change. In contrast, f is dependent on the geometry. The number of bonds carrying a given current in a square lattice, is equal to the number of bonds in each strip, multiplied by the number of strips. Thus, $f_{\text{square}} = f_{\text{strip}} + \varphi$. Thus the spectrum relative to a strip geomery can simply be obtained from figure 4, by translating the f-axis by a quantity $\varphi \approx 0.37$. Similarly, the scaling exponent of any moment computed on a strip geometry can be obtained from the one relative to the square geometry by substracting the exponent φ.

4. Conclusion

In this paper, we have addressed the transport properties of a diode network at the directed percolation threshold. We have numerically observed previously reported properties on the geometrical and electrical properties. We have also shown the multifractal nature of the current distribution, and discussed the various forms it may take depending on the overall geometry of the networks considered.

More extensive numerical work may be helpful to reach more definitive conclusions about some numerical estimates of critical exponents, although the rapid increase of computer time needed to handled this nonlinear problem may soon turn out to be prohibitive.

Acknowledgment

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