Asymptotic and exponential stability of certain third-order non-linear delayed differential equations: Frequency domain method

Anthony Uyi Afuwape *, Jairo Eloy Castellanos

Departmento de Matemáticas, Universidad de Antioquia, Calle 67, No. 53-108, Medellín AA 1226, Colombia

A R T I C L E   I N F O

Keywords:
Third order delay differential equation
Asymptotic stability
Exponential stability
Periodic solutions
Frequency domain method

A B S T R A C T

We use the frequency domain method to prove that the zero solution of certain third order nonlinear delayed differential equations is asymptotically stable, (when there is no forcing term). We also prove the existence of a bounded solution which is exponentially stable, (when there is a bounded forcing term). The situation for which the non-linear term is delayed is also proved.

© 2010 Elsevier Inc. All rights reserved.

1. Introduction

There has been much work on systems of the form

\[
\begin{align*}
\dot{Y}(t) &= (A + B)Y(t) - Q\varphi(\sigma(t)), \\
\sigma(t) &= D'Y(t),
\end{align*}
\]

(1.1)

where \(A, B, Q, D\) are \(n \times n, n \times n, n \times m\) and \(n \times m\) real matrices, with emphasis on the qualitative properties of the solutions, (\(D'\) being the conjugate transpose of matrix \(D\)).

However, for systems of the form

\[
\begin{align*}
\dot{X}(t) &= AX(t) + BX(t - \tau) - Q\varphi(\sigma(t)), \\
\sigma(t) &= C_1X(t) + C_2X(t - \tau),
\end{align*}
\]

(1.2)

where \(C_1, C_2\) are \(n \times m\) matrices, (\(C_1', C_2'\) being conjugate transpose of matrices \(C_1, C_2\) respectively), when they are generalized to third order nonlinear equations with delays, there has been not much work in the literature while using the frequency domain techniques, especially for similar qualitative properties whenever the parameter \(\tau\) is sufficiently small and the sector conditions

\[
0 \leq \frac{\varphi_j(\sigma_{1j}) - \varphi_j(\sigma_{2j})}{\sigma_{1j} - \sigma_{2j}} \leq \mu_j \quad (\sigma_{1j} \neq \sigma_{2j}),
\]

(1.3)

\((j = 1, 2, \ldots, m)\) are satisfied.

For the study of qualitative properties of systems of these types, different methods have been used: viz:- Lyapunov functionals have been constructed and used in [1,5,6,26,30,33–40]; topological degree methods were used in [7,29]; and...
Pontryagin’s principles were used in [12]. However the use of the frequency domain method has not been exploited to its full advantage by avoiding the construction of Lyapunov functionals, which remains an art. This method dates back to the works of Brockett and Willems [11], Kalman [21], Popov and Halanay [23] and Yacubovich [41]. A good account of this is recorded in [17,22], and recent works of Duan et al. [13].

Our objective in this work is to use the frequency domain method on certain third order non-linear delayed differential equations of the forms:

\[
\begin{align*}
\dot{x}''(t) &= a\dot{x}'(t) + [b_1x'(t) + b_2x'(t - \tau)] + [c_1x(t) + c_2x(t - \tau)] + h(x(t)) = p(t), \\
\dot{x}'''(t) &= a\dot{x}'(t) + [b_1x'(t) + b_2x'(t - \tau)] + [c_1x(t) + c_2x(t - \tau)] + h_1(x(t - \tau)) = p(t), \\
\dot{x}'''(t) &= a\dot{x}'(t) + [b_1x'(t) + b_2x'(t - \tau)] + [c_1x(t) + c_2x(t - \tau)] + g(x'(t)) = p(t),
\end{align*}
\]

and

\[
\dot{x}'''(t) + ax(t) + [b_1x'(t) + b_2x'(t - \tau)] + [c_1x(t) + c_2x(t - \tau)] + g_1(x'(t - \tau)) = p(t),
\]

where \(a, b_1, b_2, c_1, c_2\), are constants and \(h, h_1, g, g_1\) and \(p(t)\) are real valued continuous functions depending only on the arguments displayed. On these equations, we shall prove

(i) that the zero solution is asymptotically stable, (when there is no forcing term, i.e. \(p(t) = 0\)); and
(ii) the existence of a bounded solution which is exponentially stable (when there is a bounded forcing term, i.e. \(p(t) \neq 0\) but bounded). The results will be extensions of the works of Afuwape [3,4], applications of Gromova and Pelevina [15] and generalization of the works of [27–29]. It will also give another method of discussing the results of [26] and particular cases of [30–33] and the recent works of Tunc [34–40].

These types of equations occur in biological models as well as in physical problems (see [14] and the references in it). They also constitute equations of interest studied by different researchers recently (see [6,26–28] through [40]). These were in different forms and with particular values for the constants \(a, b_1, b_2, c_1, c_2\), and the recent work of Duan et al. [13] and the references therein), to arrive at a general theorem for systems of the form (1.2).

2. Preliminaries

First we recall that in a series of papers, in the early 70s, Barbalat [8], Barbalat and Halanay [9,10], Halanay [16–20] and Rasvan [24,25] built on the works of Brockett and Willems [11], Kalman [21], Popov and Halanay [23] and Yacubovich [41] in [22], (and the recent work of Duan et al. [13] and the references therein), to arrive at a general theorem for systems of the form (1.2).

Precisely, the following theorem was proved:

**Theorem 2.1.** Suppose that for system (1.2)

(i) the system

\[
X(t) = AX(t) + BX(t - \tau)
\]

is such that the equation

\[
\det(pI - A - Be^{-p\tau}) = 0.
\]

has all the roots \(p\) with Re\(p < 0\), (that is uniformly asymptotically stable);

(ii) the nonlinear function \(\phi(\sigma) = \text{Col}(\phi_1(\sigma_1), \phi_2(\sigma_2), \ldots, \phi_m(\sigma_m))\), satisfies

\[
\phi_j(0) = 0, \\
0 \leq \left(\frac{\phi_j(\sigma_j)}{\sigma_j}\right) \leq \mu_j \quad (\sigma_j \neq 0),
\]

for constants \(\mu_j (j = 1, 2, \ldots, m)\);

(iii) there exist a diagonal matrix \(L\) with non-negative elements, and a diagonal matrix \(R\), such that for \(\delta > 0\), the frequency domain condition

\[
G(\omega) + G^*(-\omega) \succeq \delta I > 0 \quad \text{for all real } \omega \in [-\infty, +\infty],
\]

where

\[
G(p) = LK^{-1} + (L + pR)T(p),
\]

with the transfer matrix

\[
T(p) = (C_1 + C_2 e^{-p\tau})(pI - A - Be^{-p\tau})^{-1}Q.
\]
Remark 2.1. We note that matrices $L$ and $R$ are parameters to be chosen conveniently so that the frequency domain condition (2.2) could be achieved. The choice of these parameters increases in difficulty with the number of nonlinear functions and conditions that make the characteristic matrix have roots with negative real parts.

Remark 2.2. We also note that for $m = 1$, the frequency domain condition (2.2) is a functional inequality, which can be reduced to finding a parameter $\theta$ such that

$$\frac{1}{\mu} + \Re((1 + \imath \omega \theta)Y(\imath \omega)) \geq 0$$

for all $\omega \in \mathbb{R}$.

3. Main results—asymptotic stability with $p(t) = 0$.

The following are the main results:

3.1. We shall first consider equations of the form

$$x^n(t) + ax^2(t) + [b_1x'(t) + b_2x'(t - \tau)] + [c_1x(t) + c_2x(t - \tau)] + h(x(t)) = 0. \quad (3.1)$$

Theorem 3.1. Suppose that in Eq. (3.1), the following conditions are satisfied:

(i) $a > 0; a(b_1 + b_2) - (c_1 + |c_2|) > 0; c_1 > |c_2|;
\quad a^2 > 2(b_1 + b_2); (b_1 + b_2)^2 > 2a(c_1 + |c_2|)$;

(ii) $h(0) = 0$ and for some $\delta > 0$, there exists $\mu_1 > 0$ such that

$$0 \leq \frac{h(x(t))}{x(t)} \leq \mu_1 \quad (x(t) \neq 0),$$

with

$$\mu_1 < (c_1 - |c_2|). \quad (3.4)$$

Then, the zero solution of system (3.1) is asymptotically stable for all initial functions $x_{ic}(t) = \psi(t) \in \mathbb{C}_{-1,0^+}$.

We recall from [2] an important lemma that will be useful in the proof of Theorem 3.1.

Lemma 3.1. Let $a > 0$, $ab > c$ and $c > 0$. Then the function $\Phi_1(\nu)$ defined by

$$\Phi_1(\nu) = a\nu + \frac{\nu(\nu - b)^2}{(a\nu - c)} \quad (3.5)$$

is strictly convex for $\nu > \frac{c}{a}$ and has its minimum value $\Phi_1(\nu_1)$ at $\nu_1$ with $ab > \Phi_1(\nu_1) > c$ where $\nu_1$ satisfies

$$\Psi_1(\nu) = 2a\nu^3 + (a^3 - 2ab - 3c)\nu^2 + 2c(2b - a^2)\nu + c(ac - b^2) = 0. \quad (3.6)$$

Proof of Lemma 3.1: Differentiating $\Phi_1(\nu)$ twice, we obtain

$$(a\nu - c)^2 \Phi_1''(\nu) = \Psi_1'(\nu),$$

and

$$\frac{1}{2}(a\nu - c)^3 \Psi_1''(\nu) = \nu(a\nu - c)^2 + c(b - \nu)(a\nu - c) + c(ab - c)(b - \nu).$$

Now for $\nu > \frac{c}{a}$ and with the assumption that $b > \frac{c}{a}$, we have

$$\Phi_1'(\frac{c}{a}) > 0.$$ 

Also, $\Psi_1'(\frac{c}{a}) < 0$, and $\Psi_1(b) > 0$. Hence $\nu_1$ satisfies $\frac{c}{a} < \nu_1 < b$. Moreover, $\Phi_1(\nu) > a\nu + c$, for $\nu > \frac{c}{a}$.
Proof of Theorem 3.1: Let us write (3.1) in the system form (1.2) by putting
\[
A = \begin{pmatrix} 0 & 1 & 0 \\ -c_1 & -b_1 & -a \\ 1 & 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 0 & 0 \\ -c_2 & -b_2 & 0 \end{pmatrix}; \\
C_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad C_2 \equiv 0; \quad Q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]
We assume that the characteristic equation
\[
\Delta(p) = \text{det}(pl - A - B e^{-pt}) = p^3 + ap^2 + p(b_1 + b_2 e^{-pt}) + (c_1 + c_2 e^{-pt}) = 0
\]
has solutions \( p \) with \( \Re p < 0 \).

Clearly, \( \Delta(i\omega) = \mathcal{A} - i\mathcal{B} \), where
\[
\mathcal{A} = (c_1 - a\omega^2) + c_2 \cos \omega \tau + b_2 \omega \sin \omega \tau = (c_1 - a\omega^2) + R \cos(\omega \tau - \psi_1),
\]
and
\[
\mathcal{B} = \omega(\omega^2 - b_1) - b_2 \omega \cos(\omega \tau - \psi_1) + c_2 \sin(\omega \tau - \psi_1),
\]
with \( \psi_1 = \arctan \left( \frac{c_2}{\omega} \right) \) and \( R = \sqrt{c_2^2 + b_2^2}. \)

In fact, \( |\Delta(i\omega)|^2 = (\mathcal{A})^2 + (\mathcal{B})^2 \neq 0 \) for all \( \omega \in \mathbb{R} \), since \( \mathcal{A} = 0 \) and \( \mathcal{B} = 0 \) have no common real solutions, whenever \( a(b_1 + b_2) - c_1 > |c_2| \).

Thus, \( (i\omega - A - B e^{-i\omega})^{-1} \) exists, and in fact, the transfer function is
\[
T(i\omega) = \frac{\mathcal{A} + i\mathcal{B}}{\mathcal{A}^2 + \mathcal{B}^2}.
\]

Now, if we choose \( L = 1 \) and \( R = \theta \), we have the frequency domain condition (2.2) as
\[
\frac{1}{L} + \frac{\mathcal{A} - \omega \mathcal{B}}{\mathcal{A}^2 + \mathcal{B}^2} > 0
\]
for all \( \omega \in \mathbb{R} \), with \( \frac{1}{L} = \frac{1}{\theta} - \delta \).

For \( \theta = 0 \) and \( \tau \neq 0 \), we have to show that
\[
\frac{1}{L} + \frac{\mathcal{A}}{\mathcal{A}^2 + \mathcal{B}^2} > 0,
\]
and
\[
\left[ \frac{(c_1 - a\omega^2) + c_2 \cos \omega \tau + b_2 \tau \omega \sin(\omega \tau)}{(c_1 - a\omega^2) + c_2 \cos \omega \tau + b_2 \tau \omega \sin(\omega \tau)} \right]^2 + \left[ \frac{\omega(\omega^2 - b_1) - b_2 \omega \cos(\omega \tau - \psi_1) + c_2 \sin(\omega \tau - \psi_1)}{\omega(\omega^2 - b_1) - b_2 \omega \cos(\omega \tau - \psi_1) + c_2 \sin(\omega \tau - \psi_1)} \right]^2 > 0
\]
for all \( \omega \in \mathbb{R} \).

By Lemma 3.1, using conditions (3.2), and assuming that
\( (a - b_2 \tau)(b_1 + b_2 - c_2 \tau) > c_1 - |c_2| \),
we have
\[
k < c_1 - |c_2| + \min_{\omega \tau \neq 0} \Theta_1(\omega^2),
\]
where
\[
\Theta_1(\nu) = (a - b_2 \tau)\nu + \frac{\nu(|b_1 + b_2 - c_2 \tau|^2)}{|(a - b_2 \tau)\nu - (c_1 - |c_2|)|}.
\]
For \( \theta \neq 0 \) and \( \omega \neq 0 \) inequality (3.7) can be expanded to have
\[
\omega^6 + \omega^4 \left[ a^2 - 2(b_1 - b_2 \cos \omega \tau) + (c_1 - ab_2) \tau \left( \frac{\sin \omega \tau}{\omega \tau} \right) - k \theta \right]
+ \omega^2 b_2^2 + b_2^2 - 2a(c_1 + c_2 \cos \omega \tau) + 2b_1 b_2 \cos \omega \tau
+ \tau(2b_2 c_1 - 2c_2 b_1 - k(c_2 \theta + b_2)) \left( \frac{\sin \omega \tau}{\omega \tau} \right) + k[\theta(b_1 + b_2 \cos \omega \tau - a)]
+ (c_1^2 + c_2^2 + c_1 c_2 \cos \omega \tau + kc_2 \cos \omega \tau)
> 0
\]
for all $\omega \in [-\infty, \infty]$, if
\[ k\theta < a^2 - 2(b_1 + b_2) + \tau(ab_2 - c_2). \]
Choosing
\[ \tau < \min \left\{ \frac{a}{b_2} \frac{b_1 + b_2}{c_2} \right\}, \]
the frequency domain condition (3.7) is satisfied for all $\omega \in [-\infty, \infty]$, and the conclusion of Theorem 3.1 follows from Theorem 2.1.

**Remark 3.1.** We note that the above Theorem 3.1 is an extension of the result in [4], (with $b_2 \neq 0$).

3.2. Next, let us consider the equation of the form
\[ x''(t) + ax'(t) + [b_1x'(t) + b_2x'(t - \tau)] + [c_1x(t) + c_2x(t - \tau)] + h_1(x(t - \tau)) = 0. \tag{3.8} \]
where the nonlinearity is with delay term.

**Theorem 3.2.** Suppose that in Eq. (3.8), the following conditions are satisfied:

(i) $a > 0$, $a(b_1 + b_2) - c_1 > |c_2|; (c_1 + c_2) > 0$ and $a(b_1 + b_2) < 2(c_1 + c_2)$

(ii) there exists $\mu_2 > 0$ such that
\[ 0 < \frac{b_1(x(t))}{x(t)} \leq \mu_2 \quad (x(t) \neq 0), \tag{3.9} \]
where
\[ \mu_2 < c_1 + |c_2|, 4c_2(c_2 - 1) \quad (|c_2| > 1). \tag{3.10} \]
Then, the zero solution of Eq. (3.8) is asymptotically stable for all initial function $x_0(t) = \psi(t) \in C[-\tau, 0]$. 

**Proof of Theorem 3.2:** Using the same set up as in the proof of Theorem 3.1 except that in this case $C_1 \equiv 0$; and
\[ C_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \]
we have the transfer function as
\[ \mathcal{Y}(j\omega) = \frac{(A \cos \omega \tau - B \sin \omega \tau) - i(B \cos \omega \tau + A \sin \omega \tau)}{\mathcal{A}^2 + \mathcal{B}^2}. \]
Hence the frequency domain condition (2.2) becomes,
\[ \frac{1}{\mu_2} + \frac{(A + \omega \theta B) \cos \omega \tau - (\mathcal{B} - \omega \theta A) \sin \omega \tau}{\mathcal{A}^2 + \mathcal{B}^2} > 0. \tag{3.11} \]
This gives
\[ \frac{1}{\mu_2} + \frac{2(\cos^2 \omega \tau - 1)(c_2 - b_2\theta b_2) + (b_2 + c_2\theta) \sin(2\omega \tau)}{\mathcal{A}^2 + \mathcal{B}^2} \]
\[ + \frac{[c_1 \cos \omega \tau - \omega^2(a \cos \omega \tau + b_1 \theta)] - \omega \mathcal{B}^2(1 + a \theta) - (b_1 + c_1 \theta) \sin \omega \tau}{\mathcal{A}^2 + \mathcal{B}^2} \]
\[ > 0. \]
Choose $\theta = -\frac{1}{2}$. Then, for small values of $\tau$ this will be true. In particular, if $\tau < \frac{a^2 - b_1}{2a(a\theta - \omega^2)}$. Moreover, we have for $\omega^2 = 0$, that this is valid for $\mu_2 > 0$.

The conclusion of the proof of the theorem follows the same argument, using Theorem 2.1.

**Remark 3.2.** We note that the above Theorem 3.2 is an extension of the result in [4], (with $b_2 \neq 0$).

3.3. We can now consider equation
\[ x''(t) + ax'(t) + [b_1x'(t) + b_2x'(t - \tau)] + [c_1x(t) + c_2x(t - \tau)] + g(x'(t)) = 0. \tag{3.12} \]
The nonlinearity is on the first derivative.
Theorem 3.3. Suppose that in Eq. (3.12), the following conditions are satisfied:

(i) \( a > 0; \quad \alpha (b_1 + b_2) - (c_1 + |c_2|) > 0; \quad c_1 > |c_2|; \quad a^2 > 2(b_1 + b_2); \)
\[
(b_1 + b_2)^2 > 2\alpha (c_1 + |c_2|); \tag{3.13}
\]

(ii) \( g(0) = 0 \) and for some \( \mu_3 > 0, \)
\[
0 \leq \frac{g(x'(t))}{x'(t)} \leq \mu_3 \quad (x'(t) \neq 0), \tag{3.14}
\]
such that
\[
\mu_3 < a^2 - 2(b_1 + b_2). \tag{3.15}
\]

Then, the zero solution of Eq. (3.12) is asymptotically stable for all initial conditions \( x_0(t) = \psi(t) \in C^1_{\left[-\infty, 0\right]} \).

Proof of Theorem 3.3: We now consider Eq. (1.6) (or Eq. (3.12)) in its equivalent form (1.2) by choosing matrices \( A \), \( B \), and \( Q \) as in Theorem 3.1, but with
\[
C_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad C_2 \equiv 0.
\]
In this case, the transfer function
\[
\gamma(i\omega) = C_1(A + Be^{-i\omega T} - i\omega I)^{-1}Q = \frac{i\omega \lambda(-i\omega)}{|A(i\omega)|^2},
\]
\[
\Delta(i\omega) = \det(i\omega I - A - Be^{-i\omega T}) = \lambda - i\beta \quad \text{with}
\]
\[
\lambda = [(c_1 - a^2) + c_2 \cos \omega T + b_2 \omega \sin \omega T];
\]
\[
\beta = [\omega (\omega^2 - b_1) - b_2 \omega \cos \omega T + c_2 \sin \omega T].
\]
Hence, the frequency domain condition (2.2) becomes
\[
\frac{1}{\mu_3} + \text{Re}\left\{(1 + i\omega \theta) \frac{i\omega \lambda(-i\omega)}{|A(i\omega)|^2} \right\} > 0. \tag{3.16}
\]
Evaluating this we have
\[
\frac{1}{\mu_3} - \frac{\omega \beta - \omega \theta \lambda}{\lambda^2 + \beta^2} > 0.
\]
By inversion, and re-arrangements, we find that if we choose \( \theta \geq 0 \) and \( \mu_3 < a^2 - 2(b_1 + b_2) \), this will be true for all \( \omega \in [-\infty, +\infty] \).

The conclusion follows by using Theorem 2.1.

3.4. Lastly, we shall consider equation
\[
x''(t) + ax'(t) + [b_1x'(t) + b_2x'(t - \tau)] + [c_1x(t) + c_2x(t - \tau)] + g_1(x'(t - \tau)) = 0. \tag{3.17}
\]
where the nonlinearity has delay terms.

We have the following theorem:

Theorem 3.4. Suppose that in Eq. (3.17), the following conditions are satisfied:

(i) \( a > 0; \quad \alpha (b_1 + b_2) - (c_1 + |c_2|) > 0; \quad c_1 > |c_2|; \quad a^2 > 2(b_1 + b_2); \)
\[
(b_1 + b_2)^2 > 2\alpha (c_1 + |c_2|); \tag{3.18}
\]

(ii) \( g_1(0) = 0 \) and for some \( \mu_4 > 0, \)
\[
0 \leq \frac{g_1(x'(t))}{x'(t)} \leq \mu_4 \quad (x'(t) \neq 0), \tag{3.19}
\]
such that
\[ \mu_4 < a^2 - 2(b_1 + b_2). \] (3.20)

Then, the zero solution of Eq. (3.17) is asymptotically stable for all initial function \( x_0(t) = \psi(t) \in C^2_{[-\tau, 0]} \).

**Proof of Theorem 3.4:** Setting this as a system of the form (1.2), with matrices as before, but with \( C_1 \equiv 0; C_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), we have the transfer function as

\[ \gamma(\text{i}\omega) = C_2 e^{-i\omega t} (A + Be^{-i\omega t} - i\omega l)^{-1} Q = \frac{i\omega e^{-i\omega t} A(-\text{i}\omega)}{|\Delta(\text{i}\omega)|^2}, \]

where

\[ \Delta(\text{i}\omega) = \det(\text{i}\omega l - A - Be^{-i\omega t}) = \mathcal{A} - i\mathcal{B}, \]

with

\[ \mathcal{A} = [(c_1 - a\omega^2) + c_2 \cos \omega \tau + b_2 \omega \sin \omega \tau]. \]

\[ \mathcal{B} = [c_1 c_2(\omega^2 - b_1) - b_2 \omega \cos \omega \tau + c_2 \sin \omega \tau]. \]

Hence the frequency domain condition (2.2) becomes

\[ \frac{1}{\mu_4} + \omega \left\{ \frac{(\mathcal{A} - \mathcal{B} \theta \mathcal{B}) \sin \omega \tau - (\mathcal{B} + \mathcal{A} \theta \mathcal{A}) \cos \omega \tau}{\mathcal{A}^2 + \mathcal{B}^2} \right\} > 0. \] (3.21)

Then, this will be true for all \( \omega \in [-\infty, +\infty] \) if \( \theta < 0 \) and \( \mu_4 > \frac{1}{\theta^2} \). Application of Theorem 2.1 completes the proof of the theorem.

### 4. Exponential stability

We shall now, consider the cases when there is a forcing term \( p(t) \neq 0 \) in the Eqs. (1.4) and (1.5). We shall consider the existence of a globally exponentially stable solutions which is periodic (or almost periodic) whenever the forcing term is periodic (or almost periodic). For the general situations, much literature has been devoted (see for example [17,20,24,25]). However, in applications, very few articles have been devoted to third order nonlinear equations with delays, using the frequency domain methods.

For the general system of the form:

\[ \begin{align*}
X(t) &= AX(t) + BX(t - \tau) - Q \phi(\sigma(t)) + P(t), \\
\sigma(t) &= C_1 X(t) + C_2 X(t - \tau),
\end{align*} \] (4.1)

where \( A, B \) are \( n \times n \)-matrices; \( Q, C_1, C_2 \) are \( n \times m \)-matrices; and \( P(t) \) is an \( n \)-vector, the following theorem was proved:

**Theorem 4.1.** Suppose that for system (4.1)

(i) the system

\[ X(t) = AX(t) + BX(t - \tau) \]

is such that the equation

\[ \det(pl - A - Be^{-\delta t}) = 0 \]

has all the roots \( p \) with \( \text{Re} p < 0 \); (that is uniformly asymptotically stable);

(ii) the nonlinear function \( \phi(\sigma) = \text{CoC}(\phi_1(\sigma_1), \phi_2(\sigma_2), \ldots, \phi_m(\sigma_m)) \), satisfies

\[ 0 \leq C_{ij} (\phi_i(\sigma_i)) - (\phi_j(\sigma_j)) \leq C_{ij}, \quad (\sigma_i \neq \sigma_j), \]

\[ \phi_j(0) = 0, \]

for constants \( C_{ij}, (j = 1, 2, \ldots, m) \);

(iii) there exists a diagonal matrix \( L \) with non-negative elements such that for \( \delta > 0 \), the frequency domain condition

\[ G(\text{i}\omega) + G^*(-\text{i}\omega) > \delta l \quad \text{for all real } \omega \in [-\infty, +\infty], \]

where

\[ G(p) = L K^{-1} + L T(p), \]

with

\[ \gamma(p) = (C_1 + C_2 e^{\delta t})(pl - A - Be^{-\delta t})^{-1} Q, \]

and \( K = \text{diag}(\mu_1, \mu_2, \ldots, \mu_m) \) hold.

Author's personal copy
(iv) \( P(t) \) satisfies 
\[
|P(t)| \leq \rho_0 \quad \text{for all } t.
\]

Then, system (4.1) has a unique bounded solution on \((-\infty, \infty)\), which is exponentially stable. Moreover, if \( P(t) \) is periodic (respectively almost periodic), so also is the bounded solution.

For a detailed proof of this theorem, see Rasvan VI [24,25].

4.1. Using the above Theorem 4.1, the following is true of third order equation with delay and bounded forcing term
\[
x''(t) + ax''(t) + [b_1x(t) + b_2x(t - \tau)] + |c_1x(t) + c_2x(t - \tau)| + h(x(t)) = p(t).
\] (4.5)

**Theorem 4.2.** Suppose that in Eq. (4.5), the following conditions are satisfied:

(i) \( a > 0, a(b_1 + b_2) - (c_1 + |c_2|) > 0; c_1 > |c_2|; \)

(ii) for some \( \rho_0 > 0, |p(t)| \leq \rho_0 \) for all \( t \) in \((-\infty, \infty)\); and

(iii) \( h(0) = 0 \) and for some \( \delta > 0 \), there exists \( \mu > 0 \) such that
\[
0 \leq \frac{h(x(t)) - h(\bar{x}(t))}{x(t) - \bar{x}(t)} \leq \mu
\]

satisfying
\[
\frac{1}{\mu} > \delta + \left\{\frac{a^2c_2^2(c_1 - |c_2|)}{c_1(2|c_2| - c_1)(a^2c_2^2 - a^2 + 2(b_1 + b_2))} \right\}.
\]

Then, there exists a bounded exponentially stable solution for Eq. (4.5), which is periodic (or almost periodic) whenever \( p(t) \) is periodic (or almost periodic).

**Proof of Theorem 4.2:** Let us set up (4.5) as system (4.1), with the same matrices \( A, B, C_1, C_2 \), and \( Q \) as in Theorem 3.1, but with
\[
P(t) = \begin{pmatrix} 0 & 0 \\ 0 & P_1(t) \end{pmatrix}.
\]

We have the transfer function as
\[
T(i\omega) = \frac{\mathcal{A} + i\mathcal{B}}{\mathcal{A}^2 + \mathcal{B}^2},
\]
where \( \mathcal{A}(i\omega) = \mathcal{A} - i\mathcal{B}, \)
\[
\mathcal{A} = (c_1 - a\omega^2) + c_2\cos \omega \tau + b_2\omega \sin \omega \tau = (c_1 - a\omega^2) + R\cos(\omega \tau - \psi_1),
\]
and
\[
\mathcal{B} = \omega(a^2 - b_1) - b_2\omega \cos \omega \tau + c_2\sin \omega \tau = \omega(a^2 - b_1) + R\sin(\omega \tau - \psi_1),
\]
with \( \psi_1 = \arctan(\frac{b_2}{c_2-a\omega}) \) and \( R = \sqrt{(c_2^2 + b_2^2)\omega^2}. \)

Then, the frequency domain condition (4.4) will be
\[
\frac{1}{\mu} + \frac{\left[ (c_1 - a\omega^2) + R\cos(\omega \tau - \psi_1) \right]}{(c_1 - a\omega^2) + R\cos(\omega \tau - \psi_1)} \geq \delta > 0
\]
for all real \( \omega. \)

Now, for some constant \( k > 0, \) setting \( \frac{1}{\mu} = \frac{1}{k} - \delta, \) this inequality will be valid if,
\[
G \equiv (a^2 - R\sin(\omega \tau - \psi_1))^2 + G_1(\omega) \geq 0.
\]
for all real \( \omega, \) with
\[
G_1(\omega) = \left\{a^4(a^2 - 2b_1) + \omega^2(b_2^2 - 2ac_1 - ak - 2aR\cos(\omega \tau - \psi_1)) + 2b_1R\omega R\sin(\omega \tau - \psi_1) \right. \\
+ \left. \left[ c_1^2 + kac_1 + R^2\cos^2(\omega \tau - \psi_1) + 2c_1R\omega R\cos(\omega \tau - \psi_1) + kR\cos(\omega \tau - \psi_1) \right] \right\}.
\]
Next, we note that $\Gamma_1(\omega)$ is a quadratic in $\omega^2$, and that the constant term can be rearranged as

$$(c_1 + R \cos(\omega \tau - \psi_1))(k + c_1 + R \cos(\omega \tau - \psi_1)),$$

which is positive provided $c_1 > |c_2|$. 

Now, for $\omega = 0$, $\Gamma_1 \geq 0$, provided $c_1 > |c_2|$. Also, we know that

$$\Gamma_1 \geq \left\{ \omega^2(a^2 - 2b_1) + \omega^2 \left( b_1^2 - 2ac_1 - ak - 2ac_2 \right) + (c_1^2 + kc_1 + c_2^2 - 2c_1c_2 - kc_2) \right\} > 0$$

for all real $\omega$.

But for all non-zero real values of $\omega$, we can rewrite this as

$$\omega^2(a^2 - 2b_1) + \frac{1}{\omega^2}(c_1^2 + kc_1 + c_2^2 - 2c_1c_2 - kc_2) > 2a(c_1 + |c_2|) + ak - b_1^2. \tag{4.8}$$

The minimum of the left hand side of (4.8) is in fact

$$2(a^2 - 2b_1)^{1/2}|(c_1 - c_2)(c_1 - c_2 + k)|^{1/2}.$$ 

Thus, (4.8) will hold for all non-zero real $\omega$, if for some $k$,

$$\mathcal{C}(k) \equiv 2(a^2 - 2b_1)^{1/2}|(c_1 - c_2)(c_1 - c_2 + k)|^{1/2} - \left[ 2a(c_1 + |c_2|) + ak - b_1^2 \right] > 0. \tag{4.9}$$

The unique value of $k$ for which $\mathcal{C}(k)$ is minimum occurs when

$$k = c_1(2|c_2| - c_1)(a^2 - 2b_1) + c_2(a^2 - 2b_1),$$

$$\frac{a^2 c_2^2(c_1 - |c_2|)}{a^2 c_2^2(c_1 - |c_2|)}.$$ 

For this value of $k$, the frequency domain condition (4.6), (hence (4.4)) holds.

For $\tau$ very small, we can easily analyze this to be true for all $\omega \in [-\infty, +\infty]$ using Lemma 3.1. Of course if $\tau$ is not so small, we can rearrange

$$\mathcal{A} = [(c_1 - a\omega^2) + c_2 \cos \omega \tau + b_2 \omega \sin \omega \tau] = (c_1 - a\omega^2) + c_2 \cos \omega \tau + b_2 \omega \left( \frac{\sin \omega \tau}{\omega \tau} \right) \geq c_1 - |c_2| - \omega^2 (a - b_2 \tau).$$

Then, subject to $\tau < \frac{\pi}{2\beta}$ and the use of Lemma 3.1, we can make our conclusion.

The completion of the proof of Theorem 4.2 follows from the application of Theorem 4.1.

4.2. Next, we consider equations of the form

$$x''(t) + ax'(t) + [b_1x'(t) + b_2x'(t - \tau)] + |c_1x(t) + c_2x(t - \tau)| + h_1(x(t - \tau)) = p(t). \tag{4.10}$$

**Theorem 4.3.** Suppose that in Eq. (4.10), the following conditions are satisfied:

(i) $a > 0$, $a(b_1 + b_2) - (c_1 + |c_2|) > 0$; $c_1 > |c_2|$;

(ii) for some $\rho > 0, |p(t)| \leq \rho$, for all $t$ in $(-\infty, +\infty)$; and

(iii) $h(0) = 0$ and for some $\delta > 0$, there exists $\mu > 0$ such that

$$0 \leq \frac{h_1(x(t)) - h_1(x(t + \delta t))}{x(t) - x(t)} \leq \mu$$

satisfying $\mu < c_1 - c_2$. Then, there exists a bounded exponentially stable solution for Eq. (4.10), which is periodic (or almost periodic) whenever $p(t)$ is periodic (or almost periodic).

**Proof of Theorem 4.3:**

We now consider Eq. (4.10) or its equivalent form (4.1) by choosing matrices $A, B, P$ and $Q$, as in Theorem 4.2, but with

$$C_1 \equiv 0; \quad C_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$ 

In this case, we have the transfer function as

$$\mathcal{Y}(i\omega) = C_2 e^{i\omega \tau} (A + Be^{-i\omega \tau} - io\tau)^{-1}Q = e^{i\omega \tau} A(-i\omega) \frac{e^{i\omega \tau}}{|A(i\omega)|^2} = \frac{(A \cos \omega \tau + B \sin \omega \tau) + i(B \cos \omega \tau - A \sin \omega \tau)}{a^2 + b^2}. \tag{4.11}$$
Then, the frequency domain condition becomes
\[
\frac{1}{\mu} + \frac{c_2 + (c_1 - ab_1) \cos b_1 \tau}{[c_1 - ab_1 + R \cos (\beta \tau - \psi)]^2 + [\alpha \cos \tau - b_1 + R \sin (\beta \tau - \psi)]^2} \geq \delta > 0
\]  
(4.11)
for all real \( \omega \), with \( \psi \sim \arctan \left( \frac{b_2 \omega}{c_2} \right) \) and \( R = \sqrt{(c_2^2 + b_2^2) \omega^2} \).

We observe that the only critical case to study is for when \( b_1 = \omega^2 \). That is for
\[
\frac{1}{\mu} + \frac{c_2 + (c_1 - ab_1) \cos b_1 \tau}{[c_1 - ab_1 + R \cos (\beta \tau - \psi)]^2 + [\alpha \cos \tau - b_1 + R \sin (\beta \tau - \psi)]^2} > 0,
\]  
(4.12)
where \( \delta = \frac{1}{\mu} - \delta \). This will be positive provided \( (ab_1 - c_1) > |c_2| \).

Let \( D = (ab_1 - c_1) \). Then we can write this as
\[
D^2 - D \left( c_2 + 2b_1 \cos b_1 \tau + (c_2^2 + b_2^2 b_1 + k c_2) > 0. \right.
\]  
(4.13)
Since
\[
\frac{\sin \beta \tau}{\sqrt{b_1}} \leq 1,
\]  
this can be rewritten as
\[
D^2 - D \left( c_2 + 2b_1 \cos b_1 \tau + \left( c_2^2 + b_2^2 b_1 + k c_2 \right) > 0. \right.
\]  
(4.14)
This will be valid for every \( D \) satisfying \( D > |c_2| \), provided the discriminant is negative. Thus, for every \( 0 < k < c_1 - c_2 \), this will be true if \( b_2 > 2c_2 \). The conclusion follows from Theorem 4.1.

**Remark 4.1.** We note that for the case when \( \omega^2 = \frac{c_2}{b_1} \), the conclusion trivially follows as \( c_2 + \sqrt{\frac{c_2}{b_1}} (b_1 - \frac{c_2}{b_1}) > 0 \).

**References**


