Special $\star$-Identities in group algebras and oriented group involutions

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joint work with John H. Castillo and César Polcino

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XIX Coloquio Latinoamericano de Álgebra
Pucón, IX Región, Chile
11-14 Diciembre 2012

Pucón - Chile, 2012

[1] Supported by CAPES - Brazil, CCP IME-USP and Organizing Committee XIX CLA

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After the fundamental work of Amitsur and the interest in rings with involution developed from the 1970s by Herstein and collaborators, it is natural to consider group algebras from this viewpoint.

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- Giambruno, Polcino Milies and Sehgal [GPS09], studied Lie properties in $\mathbb{F}G^+$, under *group involutions*.
- Recently, Castillo and Polcino Milies [CP12] have studied the Lie nilpotence and the Lie n-Engel properties in $\mathbb{F}G^+$ and $\mathbb{F}G^-$, under the *oriented classical involution*.

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- $\mathcal{R}^+ = \{\alpha \in \mathcal{R} : \alpha^* = \alpha\}$: **Symmetric Elements** of $\mathcal{R}$ under $*$.
- $\mathcal{R}^- = \{\alpha \in \mathcal{R} : \alpha^* = -\alpha\}$: **Skew-Symmetric Elements** of $\mathcal{R}$ under $*$.

Algebras & involutions

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Definition

A $\mathbb{F}$-algebra $\mathcal{R} \in *$-PI, if there exists a nonzero polynomial $f(x_1, x_1^*, .., x_n, x_n^*) \in \mathbb{F}\{x_1, x_1^*, x_2, x_2^*, \ldots\}$, s.t $f(a_1, a_1^*, .., a_n, a_n^*) = 0$ for all $a_1, a_2, \ldots, a_n \in R$.

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Theorem (Linearization process)

If $\mathcal{R}$ satisfies $p(x;x^*) = \alpha m(x;x^*) + \ldots$ of degree d, where $\alpha \neq 0$ and $m(x;x^*)$ is a monomial of degree d, then $\mathcal{R}$ satisfies $p(x;x^*) = \alpha x_1 x_2 \ldots x_d + q(x;x^*)$ where each monomial of $q(x;x^*)$ is of degree d, involves x_i or x_i^* but not both and $x_1 x_2 \ldots x_d$ does not occur in $q(x;x^*)$.

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Given both an $\sigma : G \to \mathcal{U}(\mathbb{F})$ and a $* : G \to G$, an **oriented group involution** *of $\mathbb{F}G$ is defined by*

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As usual, we write G^+, $\mathbb{F}G^+$ ($\mathbb{F}G^-$), the set of **symmetric (skew-symmetric) elements** of G and $\mathbb{F}G$ under $\star$, respectively.

Lie nilpotent & Lie n-Engel

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1. *A $S \subseteq \mathcal{R}$ is said to be* **Lie nilpotent** *if there exists an $n \geq 2$ such that $[a_1, \ldots, a_n] = 0$ for all $a_i \in S$. The smallest such n is called the nilpotency index of S.*

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Obviously if S is Lie nilpotent then it is Lie n-Engel for some n.

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In our situation, we obtain:

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Lemma

Let G be a group such that $|\tilde{\zeta}(G)| = \infty$. If $\alpha \in \mathbb{F}G$ is such that $\left(\sigma(z)z^{-1}z^* - 1\right)\alpha = 0$, for all $z \in \zeta$, then $\alpha = 0$.

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Theorem

Let G be a group such that $|\tilde{\zeta}(G)| = \infty$. Then, $\mathbb{F}G^-$ or $\mathbb{F}G^+$ is Lie nilpotent of index n iff $\mathbb{F}G$ is Lie nilpotent of index n and, so G is nilpotent and p-abelian.

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Let $\mathbb{F}$ *be a field and* G *a group s.t* $|\tilde{\zeta}(G)| = \infty$. *If* $\mathbb{F}G \in *$*-PI,* f*, which is* $*$*-linear in some* x_i*, then* $\mathbb{F}G$ *satisfies* f_1*, the sum of all monomials of* f *containing* x_i *(and not* x_i^**).*

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Groups without elements of order 2

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Proposition

Let G be a group without elements of order 2 and $\mathrm{char}(\mathbb{F}) \neq 2$. Assume that $\mathbb{F}G^+$ or $\mathbb{F}G^-$ is Lie nilpotent. If the center of G has a non-symmetric non-trivial p'-element, then G is p-abelian.

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- $rr^* = r^*r$ for all $r \in \mathcal{R}$. Such a ring we call **normal**.

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2. *Both* N *and* G *are SLC-groups,* $G = N \times_\zeta C_G(N)$*, where* $C_G(N) \in \mathcal{A}$*,* $g^* = sg$ *for* $g \notin N$ *and* $*$ *is canonical on* N*.*

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Special $\star$-Identities in group algebras and oriented group involutions

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Further results

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We give some results where the involution of G is arbitrary. We highlight that some previous results from [GPS09], cannot be extended with a nontrivial σ.

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Lemma

Let G be a group without elements of order 2, $char(\mathbb{F}) > 2$. If $\mathbb{F}G^+$ is Lie n-Engel, then P is a subgroup. Moreover, G/P is abelian or N/P is abelian and $(G \setminus N)/P \subseteq (G/P)^+$.

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Now an oriented version of Lemma 2.8 in [GPS09]:

Lemma

Let $\mathbb{F}$ be a field of $car(\mathbb{F}) > 2$. If $\mathbb{F}G^+$ is Lie n-Engel under oriented group involution $\star$ and, if $g \in N^+$, then $g^{p^m} \in \zeta(G)$, for some m.

Proposition

Let G be a finite group of odd order, $\mathbb{F}$ a field of car$(\mathbb{F}) > 2$, $*$ an involution on G and σ an orientation

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A result showing the impossibility of $\sigma \not\equiv 1$:

Theorem

Let G be a finite group of even order. Assumme that G/P is abelian. If $(FG)^+$ is Lie n-Engel, then N is nilpotent. Moreover, if $\zeta(G) = 1$, then $G \cong P \rtimes \{g \in G : \sigma(g) = 1 \text{ e } g^2 = 1\}$.

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