# Oriented group involutions in group algebras

**Alexander Holguín Villa** [1]

To Prof. César Polcino Milies on the occasion of his 70th birthday.

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# Content

# Algebras, groups & involutions



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**Definition**

*An $\mathbb{F}$-algebra $\mathcal{R} \in \star$-PI, if there exists a nonzero polynomial $f(x_1, x_1^\star, .., x_n, x_n^\star) = f(x; x^\star) \in \mathbb{F}\{x_1, x_1^\star, x_2, x_2^\star, ...\}$, s.t $f(a_1, a_1^\star, .., a_n, a_n^\star) = f(a; a^\star) = 0$ for all $a_1, a_2, ..., a_n \in R$.*

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# A famous result



Let $\mathcal{R}^+ = \{\alpha \in \mathcal{R} : \alpha^\star = \alpha\}$ and $\mathcal{R}^- = \{\alpha \in \mathcal{R} : \alpha^\star = -\alpha\}$ be the sets of **symmetric and skew elements** of $\mathcal{R}$ under $*$ respectively.

We are going to denote by $\mathcal{U}(\mathcal{R})$ the group of units of $\mathcal{R}$ and by $\mathcal{U}^+(\mathcal{R}) := \mathcal{U}(\mathcal{R}) \cap \mathcal{R}^+$ the set of **symmetric units**.

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**Theorem (Amitsur 1968)**

*If $\mathcal{R}$ satisfies a P.I of the form $p(x; x^\star) = 0$ of degree $d$, then $\mathcal{R}$ satisfies a P.I in the usual sense. In particular, if $\mathcal{R}^+$ or $\mathcal{R}^-$ is P.I, then $\mathcal{R}$ is P.I.*

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- $\mathbb{F}G^+ = \{\alpha \in \mathbb{F}G : \alpha^* = \alpha\}$: **Symmetric Elements** *of* $\mathbb{F}G$ *under* $*$.

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- $\mathbb{F}G^- = \{\alpha \in \mathbb{F}G : \alpha^* = -\alpha\}$: **Skew-Symmetric Elements** *of* $\mathbb{F}G$ *under* $*$.

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## Definition

1. *A $S \subseteq \mathcal{R}$ is said to be* **Lie nilpotent** *if there exists an $n \geq 2$ such that $[a_1, \dots, a_n] = 0$ for all $a_i \in S$. The smallest such $n$ is called the nilpotency index of $S$.*
2. *For a positive integer $n$, we say that $S$ is* **Lie $n$-Engel** *if*

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**Obviously** *if $S$ is Lie nilpotent then it is Lie $n$-Engel for some $n$.*

# Known results



Assume that $\mathbb{F}G$ is endowed with a $\mathbb{F}$-linear involution. If $\mathbb{F}G^+$ and/or $\mathbb{F}G^-$ satisfies a Lie identity, what can you say about $\mathbb{F}G$?

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- Giambruno and Sehgal, in [GS93];
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- Giambruno, Polcino Milies and Sehgal [GPS09], studied Lie properties in $\mathbb{F}G^+$;
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- Recently, Castillo and Polcino Milies [CP12] have studied the Lie nilpotence and the Lie $n$-Engel properties in $\mathbb{F}G^+$ and $\mathbb{F}G^-$, (**oriented classical involutions**).

# Notation



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As usual, we write $G^+$, $\mathbb{F}G^+$ ($\mathbb{F}G^-$), the set of **symmetric (skew-symmetric) elements** of $G$ and $\mathbb{F}G$ under $\circledast$, respectively.

# Lie properties

# Groups without elements of order 2



Let $\zeta(G)$ be the center of $G$ and $\tilde{\zeta}(G) = \{z^{-1}z^* : z \in \zeta(G)\}$:

**Theorem (H., 2013)**

*Let $G$ be a group such that $|\tilde{\zeta}(G)| = \infty$. Then, $\mathbb{F}G^-$ or $\mathbb{F}G^+$ is Lie nilpotent of index $n$ iff $\mathbb{F}G$ is Lie nilpotent of index $n$.*

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**Proposition (H., 2013)**

*Let $G$ be a group without elements of order 2 and $\mathrm{char}(\mathbb{F}) \neq 2$. Assume that $\mathbb{F}G^+$ or $\mathbb{F}G^-$ is Lie nilpotent. If the center of $G$ has a non-symmetric non-trivial $p'$-element, then $G$ is $p$-abelian.*

# Normality & Lie $n$-Engel properties





Suppose that $\mathbb{F}G$ is a **normal** group algebra, i.e.,

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1. *$\mathbb{F}G^{+}$ is Lie n-Engel for some n (or Lie nilpotent);*
2. *$\mathbb{F}G^{+}$ is a subring in $\mathbb{F}G$;*
3. *$\mathbb{F}G$ is a normal group algebra.*

# Further results



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**Theorem (Castillo & H., 2013)**

*Let $\mathbb{F}$ be a field of char$(\mathbb{F}) > 2$, $G$ a group with involution $*$ and $\sigma \not\equiv 1$ an orientation. Suppose that $\mathbb{F}G^+$ is Lie n-Engel under $\circledast$. Then the following are hold:*

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1. *If $G$ is a torsion group and $\exp(G)$ is odd, then $P$ is a subgroup. Moreover, $G/P$ is abelian or $N/P$ is abelian and $(G \setminus N)/P \subseteq (G/P)^+$.*
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3. *If $g \in N^+$, then $g^{p^m} \in \zeta(G)$, for some $m$.*

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# References




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# Thanks for your attention!!