Group identities in group algebras and oriented group involutions

Alexander Holguín Villa [1]
e-mail: aholguin@uis.edu.co

Escuela de Matemáticas - UIS
Bucaramanga - Colombia

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Content

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After the fundamental work of Amitsur and the interest in rings with involution developed from the 1970s by Herstein and collaborators, it is natural to consider group algebras from this viewpoint.

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Conjecture (**Hartley's Conjecture, 1980**)

Let G be a torsion group and $\mathbb{F}$ a field. If $\mathcal{U}(\mathbb{F}G)$ satisfies a group identity, then $\mathbb{F}G$ satisfies a polynomial identity, [Lee10, Section 1.1].

Algebras, groups & involutions

Let $\mathbb{F}$ a field and $\mathcal{R}$ an $\mathbb{F}$-algebra with involution $\star$[2] s.t $\mathbb{F}^{\star} \subseteq \mathbb{F}$ and let $X = \{x_1, x_2, ...,\}$ be a fixed countable infinite set:

[2] $\star$ **is an anti-automorphism of $\mathcal{R}$ of order 2.**

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It is possible define...

$$\mathbb{F}\{x_1, x_2, ...,\} \underbrace{\Longrightarrow}_{\text{Renumbering we obtain}} \mathbb{F}\{x_1, x_1^{\star}, x_2, x_2^{\star}, ...\}$$

$$\langle x_1, x_2, ...\rangle \underbrace{\Longrightarrow}_{\text{Renumering again...}} \langle x_1, x_1^{\star}, x_2, x_2^{\star}, ...\rangle$$

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by setting $x_1^\star = x_2, x_3^\star = x_4$, and so forth.

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Definition

An $\mathbb{F}$-algebra $\mathcal{R} \in \star$-PI, if there exists a nonzero polynomial $f = f(x; x^\star) = f(x_1, x_1^\star, .., x_n, x_n^\star) \in \mathbb{F}\{x_1, x_1^\star, x_2, x_2^\star, ...\}$, such that

$$f(a_1, a_1^\star, .., a_n, a_n^\star) = f(a; a^\star) = 0 \textit{ for all } a_1, a_2, ..., a_n \in R.$$

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A famous result

Let $\mathcal{R}^+ = \{\alpha \in \mathcal{R} : \alpha^\star = \alpha\}$ and $\mathcal{R}^- = \{\alpha \in \mathcal{R} : \alpha^\star = -\alpha\}$ be the sets of **symmetric and skew elements** of $\mathcal{R}$ under $*$ respectively.

We are going to denote by $\mathcal{U}(\mathcal{R})$ the group of units of $\mathcal{R}$ and by $\mathcal{U}^+(\mathcal{R}) := \mathcal{U}(\mathcal{R}) \cap \mathcal{R}^+$ the set of **symmetric units**.

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A classical result of Amitsur [Her76, Theorem 6.5.1] says:

Theorem (**Amitsur 1968**)

If $\mathcal{R}$ satisfies a P.I of the form $p(x; x^\star) = 0$ of degree d, then $\mathcal{R}$ satisfies a P.I in the usual sense. In particular, if $\mathcal{R}^+$ or $\mathcal{R}^-$ is P.I, then $\mathcal{R}$ is P.I.

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1. To know the extent to which the properties of the symmetric (skew-symmetric) elements determine the properties of the whole group algebra.

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Main Questions

1. To know the extent to which the properties of the symmetric (skew-symmetric) elements determine the properties of the whole group algebra.

2. To know the extent to which the properties of the symmetric units determine either the properties of the whole unit group or the properties of the whole group algebra.

Affirmative answers to Hartley's Conjecture

Let $\mathbb{F}$ be an infinite field (or ring) and G a group:

- Giambruno, Jespers and Valenti, in [Lee10, Section 1.2] (**Semiprime**).
- Giambruno, Sehgal and Valenti (see [Lee10, Section 1.2]).
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Remark

- *If G is finite, then $\mathbb{F}G$ always is PI, but*
- *If $char(\mathbb{F}) = 0$, then $\mathcal{U}(\mathbb{F}G)$ is GI $\Leftrightarrow$ G is abelian.*

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- Giambruno, Sehgal and Valenti, in [GSV98]. (**Classical involution**).
- Dooms and Ruiz, in [DMR07] - **Regular group algebras**.
- Giambruno, Polcino Milies and Sehgal, [GPS09i]. (**Group involution**).

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Some lemmata

Lemma

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1. *Suppose that R is finite dimensional and $\mathcal{U}^+(R)$ is GI. Then R is a direct sum of simple algebras of dimension at most four over their centers and the symmetric elements R^+ are central in R, i.e., [GPS09i]*

$$A \cong D_1 \oplus D_2 \oplus \ldots \oplus D_k \oplus M_2(\mathbb{F}_1) \oplus M_2(\mathbb{F}_2) \oplus \ldots \oplus M_2(\mathbb{F}_l).$$

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2. *Suppose one of the following conditions holds, [DMR07]:*
   - *K is uncountable,*
   - *A has no simple components that are non-commutative division algebras other than quaternion algebras.*

Then $\mathcal{U}^+(A) \in GI$ if and only if A^+ is central in A.

Notation

Definition

Given both an orientation $\sigma : G \to \{\pm 1\}$ *and a group involution* $* : G \to G$, *an* **oriented group involution** *of* $\mathbb{F}G$ *is defined by*

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Group algebras and regularity

R with 1_R is said to be (von Neumann) regular if for any $x \in R$ there exists an $y \in R$ such that $xyx = x$.

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(**Villamayor-1959**) $\mathbb{F}G$ is regular if and only if G is locally finite and has no elements of order p in case $char(\mathbb{F}) = p$.

Group involution

Lemma (Dooms & Ruiz - 2007)

*Let $\mathbb{F}$ be an infinite field with char$(\mathbb{F}) \neq 2$ and let G be a non-abelian group such that $\mathbb{F}G$ is regular. Let $*$ be an involution on G. Suppose one of the following conditions, (**C**), holds:*

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(i) *$\mathbb{F}$ is uncountable,*

(ii) *All finite non-abelian subgroups of G which are $*$-invariant have no simple components in their group algebra over $\mathbb{F}$ that are non-commutative division algebras other than quaternion algebras.*

Then $\mathcal{U}^+(\mathbb{F}G) \in GI \Leftrightarrow G$ is an SLC-group with canonical involution.

Regular case

Theorem (**H., 2013**)

*Let $\mathbb{F}$ be an infinite field with $\mathrm{char}(\mathbb{F}) \neq 2$ and let G be a non-abelian group such that $\mathbb{F}G$ is regular. Let $\sigma : G \to \{\pm 1\}$ be a nontrivial orientation and an involution $*$ on G. Suppose one of the conditions (**C**) above holds:*

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Then $\mathcal{U}^+(\mathbb{F}G) \in GI$ if and only if one of the following conditions holds:

1. *$N = ker(\sigma)$ is an abelian group and $(G \setminus N) \subset G^+$;*
2. *G and N have the LC-property, and there exists a unique nontrivial commutator s such that the involution $*$ is given by*

$$g^* = \begin{cases} g, & \text{if } g \in N \cap \zeta(G) \text{ or } g \in (G \setminus N) \setminus \zeta(G); \\ sg, & \text{if otherwise.} \end{cases} \quad (1)$$

Non-regular case

Theorem (H., 2013)

*Let $g \mapsto g^*$ be an involution on a locally finite group G, $\sigma : G \to \{\pm 1\}$ a nontrivial orientation with $N = ker(\sigma)$ and $\mathbb{F}$ an infinite field with $char(\mathbb{F}) = p \neq 2$. Suppose that $\mathcal{U}^+(\mathbb{F}G) \in GI$ and that one of the (**C**) above holds:*

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Then we have that

1. $\overline{G} = G/P$ *is abelian, or*
2. $\overline{N} = N/P = ker(\overline{\sigma})$ *is abelian and* $(\overline{G} \setminus \overline{N}) \subset \overline{G^+}$, *or*
3. $\overline{G}$ *and* $\overline{N}$ *have the LC-property and there exists a unique non-trivial commutator* $\overline{s}$ *such that the involution* $\overline{*}$ *in* $\overline{G}$ *is given by*

$$\overline{g^*} = \begin{cases} \overline{g}, & \text{if } \overline{g} \in \overline{N} \cap \zeta(\overline{G}) \text{ or } \overline{g} \in (\overline{G} \setminus \overline{N}) \setminus \zeta(\overline{G}); \\ \overline{sg}, & \text{if otherwise.} \end{cases} \quad (2)$$

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(i) $C_G(g) = \{h \in G : hg = gh\}$: Centralizer of $g \in G$,

(ii) $\Phi(G) = \{g \in G : [G : C_G(g)] < \infty\}$, $\Phi_p = \langle P \cap \Phi \rangle$: **FC**-subgroup,

(iii) $\eta(\mathbb{F}G)$: Prime radical.

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Then $\mathcal{U}^+(\mathbb{F}G) \in GI$ if and only if P is a finite normal subgroup and G/P is abelian or G/P and N/P are as in the Theorem of the **regular case**.

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Obrigado