Oriented group involutions in group algebras. A survey

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Dedicated to Professor César Polcino Milies on the occasion of his 70th birthday.

Abstract

Let $\circledast : \mathbb{F}G \to \mathbb{F}G$ denote the involution obtained as a linear extension of an involution of G, twisted by the homomorphism $\sigma : G \to \{\pm 1\}$. In this survey we gather some results concerning to the Lie properties of symmetric and skew-symmetric elements and the corresponding group identities satisfied by the set of symmetric units, and when these identities determine the structure of the whole group algebra $\mathbb{F}G$ (resp. unit group $\mathcal{U}(\mathbb{F}G)$).

1 Introduction

Let $\mathbb{F}G$ denote the group algebra of the group G over the field $\mathbb{F}$ with char$(\mathbb{F}) \neq 2$. Any involution $* : G \to G$ can be extended $\mathbb{F}$-linearly to an algebra involution $* : \mathbb{F}G \to \mathbb{F}G$. A natural involution on G is the so-called *classical involution*, which maps $g \in G$ to g^{-1}.

Let $\sigma : G \to \{\pm 1\}$ be a group homomorphism (called an orientation). If $* : G \to G$ is a group involution, an *oriented group involution* of $\mathbb{F}G$ is defined by

$$\alpha = \sum_{g \in G} \alpha_g g \mapsto \alpha^{\circledast} = \sum_{g \in G} \alpha_g \sigma(g) g^*. \tag{1}$$

Notice that, when σ is nontrivial, char$(\mathbb{F})$ must be different from 2. It is clear that, $\alpha \mapsto \alpha^{\circledast}$ is an involution in $\mathbb{F}G$ if and only if $gg^* \in N = ker(\sigma)$ for all $g \in G$.

In the case that the involution on G is the classical involution, $g \mapsto g^{-1}$, the map $\circledast$ is precisely the oriented involution introduced by S. P. Novikov (1970) in the context of K-theory, see [36].

We write $\mathbb{F}G^+ = \{\alpha \in \mathbb{F}G : \alpha^{\circledast} = \alpha\}$ and $\mathbb{F}G^- = \{\alpha \in \mathbb{F}G : \alpha^{\circledast} = -\alpha\}$ for the set of symmetric and skew-symmetric elements of $\mathbb{F}G$ under $\circledast$, respectively and, let $\mathcal{U}^+(\mathbb{F}G)$ denote the set of $\circledast$-symmetric units, i.e., $\mathcal{U}^+(\mathbb{F}G) = \{\alpha \in \mathcal{U}(\mathbb{F}G) : \alpha^{\circledast} = \alpha\}$.

Let R be an $\mathbb{F}$-algebra. Recall that a subset S of R satisfies a polynomial identity ($S \in$ PI or S is PI) if there exists a nonzero polynomial $f(x_1, x_2, .., x_n)$ in the free associative

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$\mathbb{F}$-algebra $\mathbb{F}\{X\}$ on the set countably infinite of non-commuting variables $X = \{x_1, x_2, ...\}$ such that $f(s_1, s_2, ..., s_n) = 0$ for all $s_i \in S$. For instance, R is commutative if it satisfies the polynomial identity $f(x_1, x_2) = x_1x_2 - x_2x_1$.

The conditions under which $\mathbb{F}G$ satisfies a polynomial identity were determined in classical results due to Isaacs and Passman summarized in the following:

Recall that, for a prime p, a group G is called p-abelian if G', the commutator subgroup of G, is a finite p-group and 0-abelian means abelian.

Theorem 1.1. ([37, corollaries 3.8 and 3.10, p. 196-197]) *Let $\mathbb{F}$ be a field of characteristic p and G a group. Then $\mathbb{F}G$ satisfies a PI if and only if G has a p-abelian subgroup of finite index.*

Given an associative ring R, we define the Lie product by $[x_1, x_2] = x_1x_2 - x_2x_1$ and, we can extended it recursively via $[x_1, ..., x_n, x_{n+1}] = [[x_1, ..., x_n], x_{n+1}]$.

Let S be a subset of R. We say that S is *Lie nilpotent* if there exists an integer $n \geq 2$ such that $[a_1, ..., a_n] = 0$ for all $a_i \in S$. The smallest such n is called the index of nilpotency of S. For a positive integer n, we say that S is Lie n-Engel if

$$[a, \underbrace{b, ..., b}_{n \text{ times}}] = 0$$

for all $a, b \in S$. Obviously, if S is Lie nilpotent, then it is Lie n-Engel for some n.

We will be interested in the group algebra $\mathbb{F}G$ and the set of symmetric (skew-symmetric) elements $\mathbb{F}G^+$ ($\mathbb{F}G^-$); indeed, Lie nilpotent and Lie n-Engel group algebras (special PI algebras) have been the subject of a good deal of attention. In the early 70's, as a consequence of Theorem 1.1, Passi, Passman and Sehgal characterized when a group algebra is Lie nilpotent. Sehgal determined necessary and sufficient conditions for a group algebra to be Lie n-Engel for some n. Their results are the followings.

Theorem 1.2. ([40, theorem V.4.4]) *Let $\mathbb{F}$ be a field with* $\operatorname{char}(\mathbb{F}) = p \geq 0$, *and let G be a group. Then $\mathbb{F}G$ is Lie nilpotent if and only if G is nilpotent and p-abelian.*

Theorem 1.3. ([40, theorem V.6.1]) *Let $\mathbb{F}G$ be the group algebra of a group G over a field $\mathbb{F}$. If* $\operatorname{char}(\mathbb{F}) = 0$, *then $\mathbb{F}G$ is Lie n-Engel if and only if G is abelian. If* $\operatorname{char}(\mathbb{F}) = p > 0$, *then $\mathbb{F}G$ is Lie n-Engel if and only if G is nilpotent and G has a p-abelian normal subgroup of finite p-power index*

Let R be a ring with $1_R = 1$ and $\star$ an involution on R. Let us denote by $R^+ = \{r \in R : r^\star = r\}$ and $R^- = \{r \in R : r^\star = -r\}$ the sets of *symmetric and skew-symmetric elements* respectively of the ring R under the involution $\star$. We are going to write $\mathcal{U}(R)$ for the unit group of R and $\mathcal{U}^+(R) = \mathcal{U}(R) \cap R^+$ for the set of symmetric units. A question of general interest is which properties of R^+ or R^- can be lifted to R. A similar question may be posed for the set of the symmetric units or the subgroup that they generate, i.e., to determine the extent to which the properties of $\mathcal{U}^+(R)$ determine either the properties of the whole unit group $\mathcal{U}(R)$ or the whole ring R. After the fundamental work of Amitsur [1, 2], and the interest in rings with involution developed from the 1970s by Herstein and his collaborators, [24], it is natural to consider group algebras from this viewpoint.

If R is a $\mathbb{F}$-algebra with involution $\star$ such that $\lambda^\star = \lambda$, for all $\lambda \in \mathbb{F}$, we may define an involution on the free associative algebra $\mathbb{F}\{X\}$ (again X is a set countably infinite of non-commuting variables) by setting $x_{2n+1}^\star = x_{2n+2}$, for all $n \geq 0$. Then, after renumbering we get the free associative algebra with involution, $\mathbb{F}\{X, \star\} = \mathbb{F}\{x_1, x_1^\star, x_2, x_2^\star, ...\}$. Of course, an element in this algebra is a polynomial in the variables x_i and $x_i^\star$, which do not commute. We say that $0 \neq f(x_1, x_1^\star, ..., x_n, x_n^\star) \in \mathbb{F}\{x_1, x_1^\star, x_2, x_2^\star, ...\}$ is a $\star$-polynomial identity for a subset S of R, if $f(s_1, s_1^\star, ..., s_n, s_n^\star) = 0$ for all $s_1, s_2, ..., s_n \in S$. With this terminology in mind, the results of Amitsur can be summarized as follows.

Theorem 1.4. *Let R (with or without identity) be an $\mathbb{F}$-algebra with involution $\star$. If R satisfies a $\star$-polynomial identity, then R satisfies a (in usual sense) polynomial identity. In particular, if R^+ or R^- is PI, then R itself is PI.*

Using the theorems 1.1 and 1.4, we get in the setting of group algebras the following immediate theorem.

Theorem 1.5. *Let $\mathbb{F}$ be a field with $\text{char}(\mathbb{F}) \neq 2$ and G be a group. Let $\circledast$ an oriented group involution. Then the following statements are equivalents.*

(i) $\mathbb{F}G^+$ satisfies a PI;

(ii) $\mathbb{F}G$ satisfies a PI;

(iii) G has a p-abelian subgroup of finite index.

Notice that Amitsur's result proves the existence of an ordinary polynomial identity for the $\mathbb{F}$-algebra R however, in general, does not give any information on its degree. The reason for this failure is the following: the theorem was proved first for semi-prime rings where, through structure theory, the degree of an identity for R is well related to that of the given $\star$-identity; then the result was pushed to arbitrary rings by means of the so-called Amitsur's trick. In that procedure any information on the degree of the $\star$-identity satisfied by R is lost. This problem was solved by Bahturin, Giambruno and Zaicev in [3]. In fact, by using combinatorial methods pertaining to the asymptotic behaviour of a numerical sequence attached to the algebra R, it was shown that one can relate the degree of a $\star$-polynomial identity satisfied by R to the degree of a polynomial identity for R by mean of an explicit function.

Another question in this direction is the following: if R satisfies some special kind of $\star$-polynomial identity, what kind of ordinary identity can one get in Amitsur's result? Recalling that R^- is a Lie subalgebra of R under the bracket operation $[a, b] = ab - ba$, it is natural to ask if, in particular, the Lie nilpotence of R^- implies the Lie nilpotence (or some other special type of identity) of R. The best known result in this direction is due to Zalesskii and Smirnov.

Theorem 1.6. *Suppose that $R = \langle R^-, 1\rangle$ as a ring and that $\text{char}(R) \neq 2$. If R^- is Lie nilpotent then R is Lie nilpotent.*

In general $\langle \mathbb{F}G^-, 1\rangle \neq \mathbb{F}G$. For instance, let $\mathcal{D}_k = \langle a, b : a^k = b^2 = (ab)^2 = 1\rangle$ be a dihedral group, where $k \geq 3$. If $\text{char}(\mathbb{F}) \neq 2$, then the elements of order 1 and 2 do not appear in the support of any skew elements of $\mathbb{F}\mathcal{D}_k$. But for any i, $(a^i b)^2 = 1$, hence $\mathbb{F}\mathcal{D}_k^- = \mathbb{F}\langle x\rangle^-$, which is commutative. However, $\mathcal{D}_k$ is not nilpotent unless it is a 2-group. Thus, by Theorem 1.3, $\mathbb{F}\mathcal{D}_k$ is not Lie n-Engel.

However, from the identity $2g^2 = 2 + (g^2 - g^{-2}) + (g - g^{-1})^2$, it follows that $g^2 \in \langle \mathbb{F}G^-, 1\rangle$ for all $g \in G$. Thus if G is a finite group of odd order and $\text{char}(\mathbb{F}) \neq 2$ then $\langle \mathbb{F}G^-, 1\rangle = \mathbb{F}G$.

In this short survey we shall review some results concerning the specific PI's Lie nilpotence, Lie n-Engel and commutativity (Lie properties) in the set of symmetric and skew-symmetric elements and the corresponding group identities satisfied by the set of symmetric units, and when these identities determine the structure of the whole group algebra $\mathbb{F}G$ or the whole group of units $\mathcal{U}(\mathbb{F}G)$, respectively.

2 Lie Identities

2.1 Classical Involution

The group algebra $\mathbb{F}G$ has a natural involution given by $\alpha = \Sigma\alpha_g g \mapsto \alpha^* = \Sigma\alpha_g g^{-1}$. This involution, known as the *classical involution*, appears as a technical tool to obtain results on units in a paper of G. Higman [25]. In particular, it is used there to prove that if G is a finite abelian group, then $\mathbb{Z}G$ has non-trivial units unless either the orders of the elements of G divide four, or six, in which case $\mathbb{Z}G$ has only trivial units.

Over the past three decades, a lot of attention has been devoted to determining if Lie properties satisfied by $\mathbb{F}G^+$ or $\mathbb{F}G^-$ under the classical involution are also satisfied by the whole group algebra $\mathbb{F}G$, (see, for instance [5, 18, 32]). In case $\mathbb{F}$ is a field with $\mathrm{char}(\mathbb{F}) \neq 2$ and $\mathbb{F}G$ is semiprime, then it follows from a result of Giambruno, Polcino Milies and Sehgal [15] (Giambruno and Sehgal [18]) that $\mathbb{F}G^+$ is Lie n-Engel (resp. Lie nilpotent) if and only if it is commutative. In [5] Broche Cristo characterized in an elegant manner when for an arbitrary ring R and a group G the set $\mathbb{F}G^+$ forms a commutative ring. In order to state the Broche Cristo's results, a definition is required. Recall that a group G is a Hamiltonian 2-group if $G \cong \mathcal{Q}_8 \times E$ where $\mathcal{Q}_8 = \{x, y : x^4 = 1, y^2 = x^2, x^y = x^{-1}\}$ is the quaternion group of order 8 and E is an elementary abelian 2-group, [39, Theorem 1.8.5].

Theorem 2.1. *Let G be a group and let R be a commutative ring of characteristic different from 2. Then RG^+ is a commutative ring if and only if G is either an abelian group or a Hamiltonian 2-group.*

When $\mathrm{char}(R) = 2$, Broche Cristo obtained the following result.

Theorem 2.2. *Let G be a group and let R be a commutative ring with $\mathrm{char}(R) = 2$. Then RG^+ is a commutative ring if and only if G is either an abelian group or the direct product of an elementary abelian 2-group and a group H satisfying one of the following conditions:*

(i) H has an abelian subgroup A of index 2 and an element b of order 4 such that conjugation by b inverts each element of A;

(ii) H is either the direct product of the quaternion group of order 8 and the cyclic group of order 4, or the direct product of two quaternion groups of order 8;

(iii) H is the central product of the group $\langle x, y : x^4 = y^4 = 1, x^2 = (y, x)\rangle$ with the quaternion group of order 8, where the non-trivial element common to the two central factors is x^2y^2;

(iv) H is isomorphic to one of the groups H_{32} and H_{245}, where

$$H_{32} = \langle x, y, u : x^4 = y^4 = 1, x^2 = (y, x), y^2 = u^2 = (u, x), x^2y^2 = (u, y)\rangle,$$

y

$$H_{245} = \langle x, y, u, v : x^4 = y^4 = (u, v) = 1, x^2 = v^2 = (y, x) = (v, y), \\ y^2 = u^2 = (u, x), x^2y^2 = (u, y) = (v, x)\rangle.$$

The list of groups above was given by V. Bovdi, Kovács and Sehgal, in [4], answering the question of when the set of symmetric units of a modular group ring RG is a multiplicative group, assuming that R is a commutative ring of prime characteristic p and G is a locally finite p-group.

Broche Cristo and Polcino Milies [8] studied necessary and sufficient conditions for $\mathbb{F}G^-$ to be commutative. Their characterization is the following.

Theorem 2.3. *Let R be a commutative ring with unity with* $\mathrm{char}(R) \neq 2, 4$ *and let G be any group. Then RG^- is commutative if and only if one of the following conditions holds:*

(i) G is abelian;

(ii) $A = \langle g \in G : \mathfrak{o}(g) \neq 2 \rangle$ is a normal abelian subgroup of G;

(iii) G contains an elementary abelian 2-subgroup of index 2.

The study of Lie nilpotence of $\mathbb{F}G^+$ and $\mathbb{F}G^-$ began with Giambruno and Sehgal, in [18], where in the absence of 2-elements they proved that the Lie nilpotence of $\mathbb{F}G^+$ or $\mathbb{F}G^-$ implies the Lie nilpotence of $\mathbb{F}G$. More exactly the result is the following.

Theorem 2.4. *Let G be a group with no 2-elements and let $\mathbb{F}$ be a field with* $\mathrm{char}(\mathbb{F}) \neq 2$. *Suppose that $\mathbb{F}G^+$ or $\mathbb{F}G^-$ is Lie nilpotent.*

(i) If $\mathrm{char}(\mathbb{F}) = 0$, *then G is abelian group.*

(ii) If $\mathrm{char}(\mathbb{F}) = p > 0$, *then G is nilpotent and p-abelian.*

Later G. Lee considered the case when $\mathbb{F}G^+$ has 2-elements. He obtained different answers depending if G contains a copy of the quaternion group $\mathcal{Q}_8$ of order 8. The result can be stated in the following theorem.

Theorem 2.5. *Let $\mathbb{F}$ be a field with* $\mathrm{char}(\mathbb{F}) \neq 2$ *and let G be a group. If $\mathcal{Q}_8 \nsubseteq G$ then $\mathbb{F}G^+$ is Lie nilpotent if and only if $\mathbb{F}G$ is Lie nilpotent. In case $\mathcal{Q}_8 \subseteq G$, then $\mathbb{F}G^+$ is Lie nilpotent if and only if one of the following conditions holds:*

(i) $p = 0$ and G is a Hamiltonian 2-group;

(ii) $p > 2$ and $G \cong H \times P$, where H is a Hamiltonian 2-group and P is a finite p-group.

Using the characterization of the Lie nilpotence of $\mathbb{F}G$ given by Passi, Passman and Sehgal, Theorem 1.2 and, taking into account theorems 2.4 and 2.5, we get as an immediate consequence that the Lie nilpotence of $\mathbb{F}G^+$ is equivalent to the Lie nilpotence of whole algebra $\mathbb{F}G$.

The study of the nilpotency of the set of skew-symmetric elements of group algebras with elements of order 2 has been more complicated and took a rather long time. This study began with Giambruno and Polcino Milies, in [14]. In that paper, Giambruno and Polcino Milies exhibited a nontrivial example of a group G for which $\mathbb{F}G^-$ is commutative, so Lie nilpotent.

Lemma 2.1. *Suppose that G contains an abelian subgroup A of index 2. If either $A^2 = 1$ or $(G \setminus A)^2 = 1$, then $[\mathbb{F}G^-, \mathbb{F}G^-] = 0$.*

Moreover, in case the group algebra $\mathbb{F}G$ is semiprime, Giambruno and Polcino Milies proved that actually the example given in the last lemma is exhaustive of all possibilities. The study of this Lie identity was recently completed by Giambruno and Sehgal in [19] with the proof of the following result.

Theorem 2.6. *Let $\mathbb{F}G$ be the group algebra of a group G over a field $\mathbb{F}$ with* $\mathrm{char}(\mathbb{F}) \neq 2$ *endowed with the classical involution. Then $\mathbb{F}G^-$ is Lie nilpotent if and only one of the following conditions holds:*

(i) G has a nilpotent p-abelian subgroup H with $(G \setminus H)^2 = 1$;

(ii) *G has an elementary abelian* 2*-subgroup of index* 2*;*

(iii) *the* p*-elements of* G *form a finite normal subgroup* P *and* G/P *is an elementary abelian* 2*-group.*

We finish this section with the questions concerning to the study of Lie n-Engel property. In [30], Lee characterized this property. Again in this case, the answer for $\mathbb{F}G^+$ depends on the fact that G contains or not a copy of the quaternion group or not. More exactly the results can be given in the following theorem.

Theorem 2.7. *Let* $\mathbb{F}$ *be a field with* $\mathrm{char}(\mathbb{F}) \neq 2$ *and let* G *be a group. If* $\mathcal{Q}_8 \nsubseteq G$ *then* $\mathbb{F}G^+$ *is Lie* n*-Engel for some* n *if and only if* $\mathbb{F}G$ *is Lie* m*-Engel for some* m*. In case* $\mathcal{Q}_8 \subseteq G$*, then* $\mathbb{F}G^+$ *is Lie* n*-Engel if and only if one of the following conditions holds:*

(i) $p = 0$ *and* G *is a Hamiltonian* 2*-group;*

(ii) $p > 2$ *and* $G \cong H \times P$*, where* H *is a Hamiltonian* 2*-group and* P *is a nilpotent* p*-group of bounded exponent containing a normal subgroup* A *of finite index such that* A' *is also finite.*

In the case of the set $\mathbb{F}G^-$, in the same paper [30], Lee deals with groups without 2-elements and his answer is similar to that one from Giambruno and Sehgal with respect to Lie nilpotence property, see Theorem 2.4.

Theorem 2.8. *Let* $\mathbb{F}G$ *be the group algebra of a group* G *with no* 2*-elements over a field* $\mathbb{F}$ *of characteristic different from* 2 *endowed with the classical involution. Then* $\mathbb{F}G^-$ *is Lie* n*-Engel for some* n*, if and only if* $\mathbb{F}G$ *is Lie* m*-Engel, for some* m*.*

2.2 Group Involution

Recently, there has been a considerable amount of work on involutions of $\mathbb{F}G$ other than the classical involution. Let $* : G \to G$ be a function satisfying $(gh)^* = h^*g^*$ and $(g^*)^* = g$ for all $g, h \in G$. Extending it $\mathbb{F}$-linearly, we obtain an involution on $\mathbb{F}G$, the so-called induced involution (obviously, the classical involution is the one induced from $g \mapsto g^{-1}$ on G). In particular, Jespers and Ruiz Marín in [28] gave a characterization for the commutativity of the set of symmetric elements $\mathbb{F}G^+ = \{\alpha \in \mathbb{F}G : \alpha^* = \alpha\}$ with respect to the induced involution $*$.

In order to state the result due to Jespers and Ruiz Marín, we need some definitions. Let $\zeta(G) = \zeta$ denote the center of the group G. We recall that an LC-group (short for "limited commutativity") is a nonabelian group G such that if $g, h \in G$ satisfy $gh = hg$, then at least one element of $\{g, h, gh\}$ must be central. These groups were introduced by Goodaire. By Goodaire et al. [23, Proposition III.3.6], a group G is an LC-group with a unique nonidentity commutator s (obviously it has order 2) if and only if $G/\zeta(G) \cong C_2 \times C_2$. If G has an involution $*$, then we say that G is an special LC-group, or SLC-group, if it is an LC-group, it has an unique nonidentity commutator s, and for all $g \in G$ we have $g^* = g$ if $g \in \zeta(G)$ and, otherwise, $g^* = sg$. It is easy to see that if $*$ is the classical involution, the SLC-groups are precisely the Hamiltonian 2-groups. The result proved by Jespers and Ruiz Marín is the following.

Theorem 2.9. *Let* R *be a commutative ring with* $\mathrm{char}(R) \neq 2$*,* G *a non-abelian group with an involution* $*$ *which is extended* R*-linearly to* RG*. Then the following are equivalent:*

(i) RG^+ *is commutative;*

(ii) $RG^+ = \zeta(RG)$,

(iii) G *is an SLC-group.*

Notice that if $*$ is the classical involution on G, we obtain the characterization of Broche Cristo, Theorem 2.1, when RG^+ form a ring or equivalently when RG^+ is commutative.

The Theorem by Jespers and Ruiz Marín is particularly helpful in combination with the following results due to Giambruno, Polcino Milies and Sehgal in [15] (the proof for R^- is essentially identical), in their study about the conditions under which $\mathbb{F}G^+$ is Lie nilpotent and Lie n-Engel.

Lemma 2.2. *(i) [15, Lemma 2.4] Let R be a semiprime ring with involution such that $2R = R$. If R^+ (resp. R^-) is Lie n-Engel, then, $[R^+, R^+] = 0$ (resp. $[R^-, R^-] = 0$) and R satisfies St_4 the standard identity on 4 noncommuting variables.*

(ii) [15, Proposition 3.2] Suppose that $\mathrm{char}(\mathbb{F}) = p > 2$, *and $\mathbb{F}G^+$ is Lie n-Engel. Then the p-elements of G form a (normal) subgroup of G.*

We recall that a ring is said to be semiprime if it has no nonzero nilpotent ideals. The semiprime group algebras were classified in a classical result due to Passman. Let $C_G(g)$ be the centralizer of g in G. Denote by $\Phi(G) = \{g \in G : [G : C_G(g)] < \infty\}$ the **FC** centre of G and by $\Phi_p(G) = \langle P \cap \Phi(G) \rangle$ the subgroup generated by the p-elements in $\Phi(G)$.

Lemma 2.3. [37, theorems 4.2.12 and 4.2.13, p. 130-131] *Let G be any group. If $\mathbb{F}$ is a field of characteristic zero, then $\mathbb{F}G$ is semiprime. If* $\mathrm{char}(\mathbb{F}) = p > 0$, *then the following statements are equivalent:*

(i) $\mathbb{F}G$ *is semiprime;*

(ii) G *has no finite normal subgroups with order divisible by p; and*

(iii) $\Phi_p(G) = 1$.

Notice that by the last lemma, every group algebra $\mathbb{F}G$ with $\mathrm{char}(\mathbb{F}) = 0$ is semiprime. Thus, Lemma 2.2 solves this case completely, i.e., it completely determines when $\mathbb{F}G^+$ is Lie n-Engel or Lie nilpotent. By Lemma 2.2(*ii*) if $\mathrm{char}(\mathbb{F}) = p > 2$ the set P of the p-elements of G is a normal subgroup. Also, P is $*$-invariant. Thus, if $\mathbb{F}G^+$ is Lie n-Engel, then so is $F(G/P)^+$. Since G/P has no p-elements, by Lemma 2.2(*i*), $\mathbb{F}(G/P)^+$ is commutative and therefore by result of Jespers and Ruiz Marín, G/P is either abelian or an LC-group with a unique nonidentity commutator.

In [15], Giambruno, Polcino Milies and Sehgal extended the results of the theorems 2.5 and 2.7 in this setting, i.e., they showed that in absence of 2-elements if $\mathbb{F}G^+$ is Lie n-Engel for some n (resp. Lie nilpotent), then $\mathbb{F}G$ is Lie m-Engel for some (resp. Lie nilpotent). This work was completed by Lee, Sehgal and Spinelli in [33]. The results are the following.

Theorem 2.10. [33, theorems 1 and 2] *Let G be a group with an involution $*$, $\mathbb{F}$ a field with* $\mathrm{char}(\mathbb{F}) = p > 2$. *Suppose that $\mathbb{F}G$ is not Lie nilpotent (resp. m-Engel for any m). Then the following conditions holds:*

(i) $\mathbb{F}G^+$ *is Lie nilpotent if and only if G is nilpotent, and G has a finite normal $*$-invariant p-subgroup N such that G/N is an SLC-group.*

(ii) $\mathbb{F}G^+$ *is Lie n-Engel if and only if G is nilpotent, G has a p-abelian $*$-invariant normal subgroup A of finite index, and G has a normal $*$-invariant p-subgroup N of bounded exponent such that G/N is an SLC-group.*

The skew-symmetric elements of $\mathbb{F}G$ under group involutions also have been considered. In [6], Broche Cristo et al. established the conditions under which $\mathbb{F}G^-$ is commutative. Let $R_2 = \{r \in R : 2r = 0\}$ and $G^+ = \{g \in G : g^* = g\}$ the subset of $*$-symmetric elements of G. The complete answer is the following.

Theorem 2.11. *Let R be a commutative ring. Suppose G is a non-abelian group and $*$ is an involution on G. Then, RG^- is commutative if and only if one of the following conditions holds:*

(i) $K = \langle g \in G : g \notin G^+ \rangle$ *is abelian (and thus $K \cup Kx$, where $x \in G^+$, and $k^* = xkx^{-1}$ for all $k \in K$) and $R_2^2 = \{0\}$.*

(ii) $R_2 = \{0\}$ *and G contains an abelian subgroup of index 2 that is contained in G^+.*

(iii) $\mathrm{char}(R) = 4$, $|G'| = 2$, $G/G' = (G/G')^+$, $g^2 \in G^*$ *for all $g \in G$, and G^+ is commutative in case $R_2^2 \neq \{0\}$.*

(iv) $\mathrm{char}(R) = 3$, $|G'| = 3$, $G/G' = (G/G')^+$, $g^3 \in G^*$ *for all $g \in G$.*

Just like in the case of the classical involution, work on group algebras whose skew-symmetric elements satisfy a certain Lie identity is rather complicated. Suppose that G is a torsion group with no elements of order 2 and $\mathbb{F}G$ is a semiprime. If $\mathbb{F}G^-$ is Lie nilpotent (Lie n-Engel for some n), then $\mathbb{F}G^-$ is commutative by Lemma 2.2(i) and therefore, by Theorem 2.11 (Broche Cristo et al.) either G is abelian or one of the above four conditions holds. If (i) is satisfied, then $[G : K] \leq 2$ (see [6, Theorem 2.5]) hence the absence of elements of order 2 rules out (i) and (ii). Obviously, (iii) is not possible. Finally, also (iv) is not possible as $\mathbb{F}G$ is semiprime. In conclusion if $\mathbb{F}G^-$ is Lie nilpotent, then G is abelian.

Giambruno, Polcino Milies and Sehgal in [17] classified the torsion groups G with no 2-elements for which $\mathbb{F}G^-$ is Lie nilpotent. The main result in that work is the following.

Theorem 2.12. *Let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) = p \neq 2$ and G a torsion group with no elements of order 2. Let $*$ be an involution on $\mathbb{F}G$ induced by an involution of G. Then $\mathbb{F}G^-$ is Lie nilpotent if and only if $\mathbb{F}G$ is Lie nilpotent or $p > 2$ and the following conditions hold:*

(i) the set P of p-elements in G is a subgroup;

(ii) $*$ is trivial on G/P;

(iii) there exist normal $*$-invariant subgroups A and B with $B \leq A$ such that B is a finite central p-subgroup of G A/B is central in G/B with both G/A and $\{a \in A : aa^* \in B\}$ is finite.

Recently Catino et al. in [11] classified the torsion groups G with no 2-elements such that $\mathbb{F}G^-$ is Lie n-Engel for some n. Their main result is the following.

Theorem 2.13. *Let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) = p \neq 2$ and G a torsion group with no elements of order 2. Let $*$ be an involution on G, and extended it $\mathbb{F}$-linearly to $\mathbb{F}G$. Then $\mathbb{F}G^-$ is Lie n-Engel for some n if and only if either*

(i) $\mathbb{F}G$ is Lie m-Engel for some m;

(ii) $p > 2$, G has a p-abelian normal subgroup of finite index and G has a normal $*$-invariant p-subgroup N of bounded exponent such that the induced involution on G/N is trivial.

2.3 Oriented Involutions

Given both a nontrivial homomorphism $\sigma : G \to \{\pm 1\}$ (called an orientation homomorphism) and an involution $* : G \to G$ with $gg^* \in N = Ker(\sigma)$, we obtain an involution $\circledast$ on $\mathbb{F}G$ via $(\sum_{g\in G} \alpha_g g)^{\circledast} = \sum_{g\in G} \alpha_g \sigma(g) g^*$. In the case that the involution on G is the classical involution, $g \mapsto g^{-1}$, the map $\circledast$ is precisely the oriented involution introduced by Novikov in the context of K-theory [36].

Recently, Castillo and Polcino Milies [10] studied the Lie nilpotence and the Lie n-Engel properties in the sets $\mathbb{F}G^+$ and $\mathbb{F}G^-$ under an *oriented classical involution*, obtaining similar results to those ones in the case of the classical involution but interesting new situations arise.

In the study of Lie nilpotence and Lie n-Engel properties in $\mathbb{F}G^+$ under the classical involution (theorems 2.5 and 2.7) it was important to study the case when $G = \mathcal{Q}_8$. The next lemma shows that $\mathbb{F}\mathcal{Q}_8^+$ is Lie n-Engel (Lie nilpotent) only when the orientation σ is trivial.

Lemma 2.4. ([10, Lemma 2.3]) *Let $\mathbb{F}G$ a group algebra with* $\mathrm{char}(\mathbb{F}) \neq 2$ *and G a group such that $\mathcal{Q}_8 \subseteq G$. Let σ a nontrivial orientation of G. If $\mathbb{F}G^+$ is Lie n-Engel for some n, then $\mathcal{Q}_8 \subseteq N$.*

In the same paper Castillo and Polcino Milies proved that if G does not contain 2-elements and $\mathbb{F}$ is a field of characteristic $p \neq 2$, then the $\circledast$-symmetric elements of $\mathbb{F}G$ are Lie nilpotent (Lie n-Engel for some n) if and only if $\mathbb{F}G$ is Lie nilpotent (Lie m-Engel for some m).

If N denotes the kernel of σ, then N is a subgroup in G of index 2. It is clear that the involution $\circledast$ coincides on the subalgebra $\mathbb{F}N$ with the algebra involution $*$. Also, we have that the symmetric elements in G, under $\circledast$, are the symmetric elements in N under $*$. If we denote the set of symmetric elements in G, under $*$, by G^+, then we can write $N^+ = N \cap G^+$. Thus, if $\mathbb{F}G^+$ or $\mathbb{F}G^-$ satisfies a Lie identity, then also $\mathbb{F}N$ satisfies a Lie identity.

If $\mathbb{F}G^+$ is Lie n-Engel (Lie nilpotent) under the oriented classical involution with $\mathcal{Q}_8 \subseteq G$, then using Lee's results (theorems 2.5 and 2.7), Castillo and Polcino Milies obtained the following characterizations.

Theorem 2.14. *Let $\mathbb{F}$ be a field of characteristic $p \neq 2$, G a group with a nontrivial orientation σ and x, y elements of G such that $\langle x, y\rangle \cong Q_8$.*

1. *[10, Theorem 4.1] Then $\mathbb{F}G^+$ is Lie n-Engel, for some $n \geq 0$ if and only if either*
   - *(i)* $\mathrm{char}(\mathbb{F}) = 0$, *$N \cong Q_8 \times E$ and $G \cong \langle Q_8, g\rangle \times E$, where $E^2 = 1$, and $g \in G \setminus N$ is such that $(g, x) = (g, y) = 1$ and $g^2 = x^2$; or,*
   - *(ii)* $\mathrm{char}(\mathbb{F}) = p > 2$, *$N \cong Q_8 \times E \times P$ where P is a nilpotent p-group of bounded exponent containing a normal p-abelian subgroup A of finite index and there exists $g \in G \setminus N$ such that $G \cong \langle Q_8, g\rangle \times E \times P$, $(g, x) = (g, y) = (g, t) = 1$ for all $t \in P$ and $g^2 = x^2$.*
2. *[10, Theorem 4.2] $\mathbb{F}G^+$ is Lie nilpotent if and only if either*
   - *(i)* $\mathrm{char}(\mathbb{F}) = 0$, *$N \cong Q_8 \times E$ and $G \cong \langle Q_8, g\rangle \times E$, where $E^2 = 1$ and $g \in G \setminus N$ is such that $(g, x) = (g, y) = 1$ and $g^2 = x^2$; or,*
   - *(ii)* $\mathrm{char}(\mathbb{F}) = p > 2$, *$N \cong Q_8 \times E \times P$, where $E^2 = 1$, P is a finite p-group and there exists $g \in G \setminus N$ such that $G \cong \langle Q_8, g\rangle \times E \times P$, $(g, x) = (g, y) = 1$ and $g^2 = x^2$.*

G.T. Lee pointed out in [32, p. 86] that results regarding Lie properties of skew-symmetric elements under the classical involution, obtained for groups with no elements of order 2, cannot be extended to groups not containing quaternions. Similar difficulties arise in the present context, [10, p. 4417].

Let $\mathcal{D}_k = \langle x, y : x^k = 1, y^2 = 1, (xy)^2 = 1 \rangle$ be the dihedral group of order $2k$. It is possible to show that the only orientation σ such that $\mathbb{F}\mathcal{D}_k^+$ is Lie nilpotent is given by $\sigma(x) = 1$ and $\sigma(y) = -1$. In this case $N = ker(\sigma) = \langle x \rangle$, $G \setminus N = \{x^i y : 0 \leq i \leq k-1\}$, $(G \setminus N)^2 = 1$ and thus $\mathbb{F}\mathcal{D}_k^+$ is commutative but $\mathbb{F}\mathcal{D}_k$ is not Lie nilpotent when $\mathrm{char}(\mathbb{F}) \neq 2$.

Finally Castillo and Polcino Milies presented results in the case when G contains no subgroup isomorphic to $\mathcal{Q}_8$ and the group algebra $\mathbb{F}G$ is semiprime. Their results are following.

Theorem 2.15. [10, theorems 5.1 and 5.2] *Let $\mathbb{F}$ be a field of characteristic $p \neq 2$ and G a group such that $\mathcal{Q}_8 \not\subseteq G$ with a nontrivial orientation σ. Suppose that $g^2 \neq 1$ for all $g \in G \setminus N$. Then $\mathbb{F}G^+$ is Lie n-Engel (resp. Lie nilpotent), for some n, if and only if $\mathbb{F}G$ is Lie m-Engel for some m (resp. Lie nilpotent).*

Remark 1. *Suppose that $(G \setminus N)^2 = 1$ in the last theorem. Then, $\mathbb{F}G^+$ is Lie nilpotent if and only if $\mathbb{F}N^+$ is Lie nilpotent.*

Proposition 2.1. [10, Proposition 5.1] *Let $\mathbb{F}$ be a field of characteristic different from 2 and G a group such that $\mathbb{F}G$ is semiprime and $\mathcal{Q}_8 \not\subseteq G$. Then $\mathbb{F}G^+$ is Lie n-Engel for some n if and only if one of the followings holds:*

(i) G is abelian;

(ii) $N = Ker(\sigma)$ is abelian and $(G \setminus N)^2 = 1$.

In [7], Broche Cristo and Polcino Milies characterized when the set RG^+ under the oriented group involution given by expression (1) is commutative, where R is a commutative ring with 1_R and G is a group. Notice that if RG^+ is commutative, then RN^+ is commutative and by Theorem 2.9, the structure of N and the action of $*$ on N are known. Regardless this it is not an easy task to describe G and the action of $*$ on G. Their characterization is the following.

Theorem 2.16. [7, theorems 2.2 and 2.3] *Let R be a commutative ring with 1_R and let G be a non-abelian group with involution $*$ and non-trivial orientation homomorphism σ. Then*

1. *RG^+ is commutative if and only if one of the following conditions holds:*

   (i) N is an abelian group and $(G \setminus N) \subset G^+$;

   (ii) G and N have the LC*-property, and there exists a unique nontrivial commutator s such that the involution $*$ is given by*

$$g^* = \begin{cases} g, & \text{if } g \in N \cap \zeta(G) \text{ or } g \in (G \setminus N) \setminus \zeta(G); \\ sg, & \text{otherwise.} \end{cases}$$

   (iii) $\mathrm{char}(R) = 4$*, G has the* LC*-property, and there exists a unique nontrivial commutator s such that the involution $*$ is the canonical involution, i.e., $*$ is given by*

$$g^* = \begin{cases} g, & \text{if } g \in \zeta(G); \\ sg, & \text{otherwise.} \end{cases} \quad (2)$$

2. If $*$ is the classical involution, $g^* = g^{-1}$, then RG^+ is commutative if and only if one of the following conditions holds:

(i) N is an abelian group and $(G \setminus N)^2 = 1$;

(ii) $N = \ker(\sigma) \cong \langle x, y : x^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1}\rangle \times E$ and

$$G \cong \langle x, y, g : x^4 = 1, x^2 = y^2 = g^2, x^y = x^{-1}, x^g = x, y^g = y\rangle \times E,$$

where E is an elementary abelian 2-group;

(iii) $\mathrm{char}(R) = 4$ and G is a Hamiltonian 2-group.

A ring R with involution $a \mapsto a^\star$ is said to be normal if $aa^\star = a^\star a$, for all $a \in R$. In [24, p. 97] Herstein studied this *special class* of rings with involution.

Holguín-Villa in his PhD thesis [26, cap. 2] characterized group algebras $\mathbb{F}G$ which are normal in regard to a group involution $*$. The results depend on whether G is abelian or an SLC-group. Using this characterization, in the same work an complete answer of when $\mathbb{F}G$ is a normal group algebra with respect to the oriented group involution $\circledast$, is obtained. The characterizations are following.

Theorem 2.17. *Let $g \mapsto g^*$ denote an involution on a group G and let $\sigma : G \to \{\pm 1\}$ be a nontrivial homomorphism with $N = ker(\sigma)$. Let $\mathbb{F}G$ denote the group algebra of the group G over the field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$. Then we have the following.*

1. [26, Teorema 2.3.1] $\mathbb{F}G$ is a normal group algebra with respect to the group involution $*$ if and only if G is either an abelian group or an SLC-group.

2. [26, Teorema 2.4.1] $\mathbb{F}G$ is a normal group algebra in regard to the oriented group involution $\circledast$ given by (1) if and only if either G is an abelian group or one of the following conditions holds:

(i) $N = ker(\sigma)$ is an abelian group and, we have that $x^* = x$ for $x \in G \setminus N$, $n^* = a^{-1}na$ for all $n \in N$ and for all $a \in G \setminus N$;

(ii) G and N have the LC-property and there exists a central element $g_0 \in G$ such that $G = N \cup Ng_0 = N \cup g_0N$, $g_0^* = sg_0$ and the involution $*$ on N is the canonical involution given by (2).

Currently, Castillo and Holguín-Villa are investigating [9, In preparation] which properties of Lie known for any involution $*$ defined on a group G and for the oriented classical involution, may be generalized to oriented group involutions, with some results already obtained.

Let $\zeta = \zeta(G)$ denote the center of group G. In [18, Corollary] it was proved that if $*$ is the classical involution and ζ^2 is infinite, and if $\mathbb{F}G^+$ or $\mathbb{F}G^-$ is Lie nilpotent of index n, then also $\mathbb{F}G$ is Lie nilpotent of index n. Also in [10, Proposition 2.1] Castillo and Polcino Milies obtained a similar result for *oriented classical involutions.* We adapted the proof of these results, to our situation.

Theorem 2.18. *Let G be a group such that $\tilde{\zeta}(G) = \left\{z^{-1}z^* : z \in \zeta(G)\right\}$ is infinite. Then, $\mathbb{F}G^-$ or $\mathbb{F}G^+$ is Lie nilpotent of index n if and only if $\mathbb{F}G$ is Lie nilpotent of index n.*

In the case in which G is a group with no 2-elements such that $\mathbb{F}G$ is semiprime, we have the following result.

Theorem 2.19. *Let $\mathbb{F}$ be a field of* $\mathrm{char}(\mathbb{F}) \neq 2$ *and let G be a group without elements of order* 2 *such that $\mathbb{F}G$ is semiprime. If $\mathbb{F}G^+$ is Lie n-Engel for some n. Then, G is abelian or $N = ker(\sigma)$ is abelian and $(G \setminus N) \subseteq G^+$. Moreover, $\mathbb{F}G$ is a normal group algebra.*

From last theorem, the main result in [7], Theorem 2.16(1), and the Theorem 2.17(2) we get immediately.

Corollary 2.1. *Let $\mathbb{F}$ be a field of* $\mathrm{char}(\mathbb{F}) \neq 2$ *and let G be a group without elements of order* 2 *such that $\mathbb{F}G$ is semiprime. Then, the following conditions are equivalent:*

(i) $\mathbb{F}G^+$ is Lie n-Engel for some n (resp. Lie nilpotent);

(ii) $\mathbb{F}G^+$ is commutative;

(iii) $\mathbb{F}G$ is a normal group algebra.

In [15, Lemma 2.8] it was proved that if $\sigma \equiv 1$ on G and $*$ is any involution on the group G and, if $\mathbb{F}G^+$ ($\mathrm{char}(\mathbb{F}) > 2$) is Lie n-Engel for some n, then for every symmetric element $g \in G$, g^{p^m} is central, for some $m > 0$. If we consider the oriented group involution $\circledast$, the result is the following.

Lemma 2.5. *Let $\mathbb{F}$ be a field with* $\mathrm{char}(\mathbb{F}) = p > 2$. *Suppose that $\mathbb{F}G^+$ is Lie n-Engel. If $g \in N^+$, then g^{p^m} is central for some m.*

We highlight that some previous results from [15], can not be extended with a nontrivial σ.

Let $G = \mathcal{D}_6 = \langle x, y : x^6 = 1 = y^2, (xy)^2 = 1 \rangle$ be the dihedral group of order 12. It is easy to see that $\zeta(\mathcal{D}_6) = \{1, x^3\}$. Let $\mathbb{F}$ a field with $\mathrm{char}(\mathbb{F}) = 3$, $*$ the classical involution with $\sigma(x) = 1$ and $\sigma(y) = -1$. As an $\mathbb{F}$-module $\mathbb{F}\mathcal{D}_6^+$ is generated by the set $\{1, x^3\} \cup \{x + x^5, x^2 + x^6\}$ and thus $\mathbb{F}\mathcal{D}_6^+$ is commutative. In this case, $\mathcal{D}'_6 = \{1, x^2, x^4\}$. Therefore, $\mathcal{D}_6/\zeta(\mathcal{D}_6) \cong \mathcal{D}_3$. Since $\zeta(\mathcal{D}_3) = \{1\}$, it follows that $\mathcal{D}_6$ is not nilpotent.

In general, we have the following.

Theorem 2.20. *Let G be a finite group of even order. Assumme that G/P is abelian. If $(FG)^+$ is Lie n-Engel, then N is nilpotent. Moreover, if $\zeta(G) = 1$, then $G \cong P \rtimes \{g \in G : \sigma(g) = 1 \ e \ g^2 = 1\}$.*

Remark 2. *If $g \in G$ is a torsion element of order odd, then $\sigma(g) = 1$. Hence, if G is a finite group of odd order then $\sigma(g) = 1$ for all $g \in G$, i.e., σ is trivial on G. Thus, if* $\mathrm{char}(\mathbb{F}) = p > 2$ *and $\mathbb{F}G^+$ is Lie n-Engel, then $\mathbb{F}G$ is Lie nilpotent (the same result in [15, Theorem 2.12]).*

For a given prime p, an element $x \in G$ will be a called a p-element if its order is a power of p. Let $P = \{x \in G : \mathfrak{o}(x) = p^k \text{ for some } k\}$ be the set of the p-elements. Using the Theorem 2.19 and [15, Proposition 3.2], we have the following.

Theorem 2.21. *Let G be a torsion group without elements of order* 2 *and $\mathbb{F}$ a field of* $\mathrm{char}(\mathbb{F}) = p > 2$. *If $\mathbb{F}G^+$ is Lie n-Engel, then P is a subgroup. Moreover, G/P is abelian or N/P is abelian and $(G \setminus N)/P \subseteq (G/P)^+$.*

Let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) \neq 2$ and let G be a group without elements of order 2. Suppose that $\mathbb{F}G^+$ is Lie n-Engel. Then $\mathbb{F}N^+$ also is Lie n-Engel and by Amitsur's theorem, Theorem 1.4, $\mathbb{F}N$ is PI. It then follows from a theorem of Passman [37, p. 196], Theorem 1.1, that N has a normal p-abelian subgroup A of finite index. We can assume A is $*$- invariant by replacing it by $A \cap A^*$; i.e., we have the following.

Theorem 2.22. *If $\mathbb{F}G^+$ is Lie n-Engel (Lie nilpotent), then there exists a normal p-abelian subgroup A of finite index of G, which is $*$-invariant and $A \subseteq N$.*

3 Group Identities

In this section we shall review some results concerning to group identities in the set of symmetric units $\mathcal{U}^+(\mathbb{F}G)$, and when these properties (identities) can be lifted to $\mathcal{U}(\mathbb{F}G)$ or force $\mathbb{F}G$ to satisfy a PI. A motivation for this study is the classical theorem of Amitsur, Theorem 1.4.

Let $\langle x_1, x_2, ...\rangle$ be the free group on a countable set of generators. A group H satisfies a group identity ($H \in$ GI or H is GI) if there exists a non-trivial reduced word $\omega(x_1, x_2, ..., x_n) \in \langle x_1, x_2, ...\rangle$ such that $\omega(h_1, h_2, ..., h_n) = 1$ for all $h_i \in H$. For instance, if we write $(x_1, x_2) = x_1^{-1}x_2^{-1}x_1x_2$ and $(x_1, x_2, ..., x_n, x_{n+1}) = ((x_1, x_2, ..., x_n), x_{n+1})$, for all $n \geq 2$, then H is abelian if it satisifies the group identity (x_1, x_2) and nilpotent if it satisfies $(x_1, x_2, ..., x_n)$, for some n.

Some time ago and with idea of to establish a connection between the additive and multiplicative structure of a group algebra $\mathbb{F}G$, Brian Hartley made the following famous conjecture:

Conjecture 3.1. (Hartley's Conjecture, 1980) *Let G be a torsion group and $\mathbb{F}$ a field. If the unit group $\mathcal{U}(\mathbb{F}G)$ of $\mathbb{F}G$ satisfies a group identity, then $\mathbb{F}G$ satisfies a polynomial identity.*

Giambruno, Jespers and Valenti [13] solved the conjecture for semiprime group rings, and Giambruno, Sehgal and Valenti [20] solved it in general for group algebras over infinite fields. By using the results of [20], Passman [38] gave necessary and sufficient conditions for $\mathcal{U}(\mathbb{F}G)$ to satisfy a group identity, when $\mathbb{F}$ infinite. Subsequently, Liu [34] confirmed the conjecture for finite fields and Liu and Passman in [35] extended the results of [38] to this case. The same question for groups with elements of infinite order was studied by Giambruno, Sehgal and Valenti in [22]. For further details about these results, see G. T. Lee [32, Chapter 1].

Let $*$ be an involution of a group G extended linearly to $\mathbb{F}G$. Then we have the analogue of Hartley's Conjecture, [41, p. 77].

Conjecture 3.2. *Let G be a torsion group and $\mathbb{F}$ a field. If the set $\mathcal{U}^+(\mathbb{F}G)$ of symmetric units satisfies a group identity, then $\mathbb{F}G$ satisfies a polynomial identity.*

In general, group identities on $\mathcal{U}^+(\mathbb{F}G)$ do not force group identities on $\mathcal{U}(\mathbb{F}G)$. In fact, it is easy to see that $\zeta(\mathcal{Q}_8) = \mathcal{Q}'_8 = \{1, x^2\}$ and thus the quaternion group $\mathcal{Q}_8$ is an SLC-group with a unique nonidentity commutator $s = x^2$. Then by Jespers and Ruiz Marín's result, Theorem 2.9, for any field $\mathbb{F}$ with $\text{char}(\mathbb{F}) = p > 2$, $\mathbb{F}\mathcal{Q}_8^+$ is commutative, hence $\mathcal{U}^+(\mathbb{F}\mathcal{Q}_8)$ is GI. However if $\mathbb{F}$ is a infinity field and G is a torsion group, then by a theorem of Passman [38], the following are equivalent:

1. $\mathcal{U}(\mathbb{F}G)$ is GI;
2. $\mathcal{U}(\mathbb{F}G)$ satisfies the group identity $(x, y)^{p^r} = 1$, for some $r \geq 0$;
3. G has a normal p-abelian subgroup of finite index and G' is a p-group of bounded exponent.

It follows that in this case, $\mathcal{U}(\mathbb{F}G)$ does not satisfy a GI.

3.1 Classical Involution

Giambruno, Sehgal and Valenti [21] showed that if G is a torsion group, $\mathbb{F}$ is infinite with $\mathrm{char}(\mathbb{F}) \neq 2$, and $\mathcal{U}^+(\mathbb{F}G)$ satisfies a group identity then $\mathbb{F}G$ is PI. They also classified groups such that $\mathcal{U}^+(\mathbb{F}G)$ is *GI*, in the case of the classical involution. Their result is the following.

Theorem 3.1. *Let $\mathbb{F}G$ be the group algebra of a torsion group G over an infinite field $\mathbb{F}$ with* $\mathrm{char}(\mathbb{F}) \neq 2$*, endowed with the classical involution.*

1. *If $car(\mathbb{F}) = 0$, $\mathcal{U}^+(\mathbb{F}G)$ satisfies a group identity if and only if G is either abelian or a Hamiltonian 2-group.*
2. *If* $\mathrm{char}(\mathbb{F}) = p > 2$*, then $\mathcal{U}^+(\mathbb{F}G)$ satisfies a group identity if and only if $\mathbb{F}G$ satisfies a polynomial identity and either $\mathcal{Q}_8 \nsubseteq G$ and G' is of bounded exponent p^k for some $k \geq 0$ or $\mathcal{Q}_8 \subseteq G$ and*
   - *(i) the p-elements of G form a (normal) subgroup P of G and G/P is a Hamiltonian 2-group;*
   - *(ii) G is of bounded exponent $4p^s$ for some $s \geq 0$.*

The last result was extended to non-torsion groups, see [41, Theorem 10], under the usual restriction for the only if part related to Kaplansky's Conjecture (the units of $\mathbb{F}G$ are trvial if G is a torsion-free group and $\mathbb{F}$ is a field). The following result goes in the direction of the Hartley's Conjecture and Theorem 3.1.

Theorem 3.2. *Let $\mathbb{F}G$ be the group algebra of a group G with an element of infinite order over an infinite field $\mathbb{F}$ of characteristic different from 2 endowed with the classical involution. If $\mathcal{U}^+(\mathbb{F}G)$ satisfies a group identity, then the set P of p-elements of G forms a normal subgroup and, if P is infinite, then $\mathbb{F}G$ satisfies a polynomial identity,*

After of the result by Giambruno, Sehgal and Valenti [21], Theorem 3.1, it was of interest to consider when $\mathcal{U}^+(\mathbb{F}G)$ satisfies special group identities. As the symmetric units do not, in general, form a group, let us state that we mean specifically that $\mathcal{U}^+(\mathbb{F}G)$ satisfies a group identity of the form $(x_1, x_2, ..., x_n) = 1$ for some $n \geq 2$.

Definition 3.1. *Let H be any group and S a subset of H. If S satisfies $(x_1, x_2, ..., x_n) = 1$, then so does $\langle S \rangle$.*

In [5] Broche Cristo showed that if G is a torsion group and R is a commutative ring of odd prime characteristic, then RG^+ is a commutative ring if and only if $\mathcal{U}^+(RG)$ is an abelian group. The same result holds if R is a field of prime characteristic p and G is a locally finite p-group (Bovdi, et al., [4]), or if R is a G-favorable domain and G is a torsion group (Bovdi, 2001, see [28, p. 728]).

Recall that the ring R is said to be G-favorable if for any $g \in G$ of finite order $\mathfrak{o}(g)$ there is a nonzero $\alpha_g \in R$ such that $1 - \alpha_g^{\mathfrak{o}(g)}$ is invertible in R. Notice that every infinite field is obviously G-favorable.

In [31, theorems 1 and 2] G. T. Lee determined the conditions in terms of the group G under which $\mathcal{U}^+(\mathbb{F}G)$ is nilpotent. These conditions dependent on whether or not $\mathcal{Q}_8$ is contained in G. The results are the following.

Theorem 3.3. [31, theorems 1 and 2] *Let $\mathbb{F}$ be a field of characteristic $p \neq 2$ and G a torsion group.*

1. *Suppose that $\mathcal{Q}_8 \nsubseteq G$. Then the following are equivalent:*

   (i) *$\mathcal{U}^+(\mathbb{F}G)$;*

   (ii) *$\mathcal{U}(\mathbb{F}G)$; and,*

   (iii) *G is nilpotent and p-abelian.*

2. *Suppose that $\mathcal{Q}_8 \subseteq G$. Then $\mathcal{U}^+(\mathbb{F}G)$ is nilpotent if and only if either*

   (i) *$p = 0$ and $G \cong \mathcal{Q}_8 \times E$, where E is an elementary abelian 2-group;*

   (ii) *$p > 2$ and $G \cong \mathcal{Q}_8 \times E \times P$, where E is an elementary abelian 2-group and P is a finite p-group.*

Suppose $\mathbb{F}$ has characteristic zero. If $\mathcal{U}^+(\mathbb{F}G)$ is nilpotent, then by Giambruno et al. [21, Theorem 7], Theorem 3.1, G is abelian or a Hamiltonian 2-group, i.e., $G \cong \mathcal{Q}_8 \times E$, where $E^2 = 1$. Conversely, if G is abelian or a Hamiltonian 2-group, then the symmetric elements in $\mathbb{F}G$ commute, and thus the characteristic zero case is done.

Comparing the last theorem with Theorem 2.5, it is clear the following result.

Corollary 3.1. *Let $\mathbb{F}G$ be the group algebra of a torsion group G over a field $\mathbb{F}$ of characteristic different from 2 endowed with the classical involution. Then $\mathcal{U}^+(\mathbb{F}G)$ is nilpotent if and only if $\mathbb{F}G^+$ is Lie nilpotent.*

In 2007 Lee, Polcino Milies and Sehgal studied the non-torsion case, see [32, Section 4.6].

3.2 Group Involution

The result proved by Jespers and Ruiz Marín, Theorem 2.9, was crucial for the classification of torsion group algebras endowed with an involution induced from an arbitrary involution on G with symmetric units satisfying a group identity. The question was originally studied by Dooms and Ruiz [12, Theorem 3.1]. They proved the following.

Theorem 3.4. *Let $\mathbb{F}$ be an infinite field with $\mathrm{char}(\mathbb{F}) \neq 2$ and let G be a non-abelian group such that $\mathbb{F}G$ is regular. Let $*$ be an involution on G. Suppose one of the following conditions holds:*

1. *$\mathbb{F}$ is uncountable,*

2. *All finite non-abelian subgroups of G which are $*$-invariant have no simple components in their group algebra over $\mathbb{F}$ that are non-commutative division algebras other than quaternion algebras.*

Then $\mathcal{U}^+(\mathbb{F}G)$ is GI *if and only if G is an SLC-group with canonical involution. Moreover, in this case $\mathbb{F}G^+$ is a ring contained in $\zeta(\mathbb{F}G)$.*

Using the last result and under some assumptions, Dooms and Ruiz proved that if $\mathcal{U}^+(\mathbb{F}G)$ satisfies a group identity then $\mathbb{F}G$ is PI, giving an affirmative answer to the Hartley's Conjecture in this setting. They also characterized the groups for which the symmetric units $\mathcal{U}^+(\mathbb{F}G)$ satisfy a group identity, when the prime radical $\eta(\mathbb{F}G)$ of $\mathbb{F}G$ is nilpotent.

Giambruno, Polcino Milies and Sehgal [16] completely characterized group algebras of torsion groups, with group involutions such that $\mathcal{U}^+(\mathbb{F}G)$ is GI. Their result is the following.

Theorem 3.5. *Let $\mathbb{F}$ be an infinite field of characteristic $p \neq 2$, G a torsion group with an involution $*$ which is extended linearly to $\mathbb{F}G$. Then we have the following:*

1. *If $\mathbb{F}G$ is semiprime then $\mathcal{U}^+(\mathbb{F}G)$ is GI if and only if G is abelian or an SLC-group.*
2. *If $\mathbb{F}G$ is not semiprime then $\mathcal{U}^+(\mathbb{F}G)$ is GI if and only if, the p-elements of G form a (normal) subgroup P, $\mathbb{F}G$ is PI and one of the following holds:*
   - *(i) G/P is abelian and G' is of bounded p-power exponent.*
   - *(ii) G/P is SLC and G contains a normal $*$-invariant p-subgroup B of bounded exponent such that P/B is central in G/B and $*$ is trivial on P/B.*

Just like in the of the classical involution case (Broche Cristo [5], Bovdi, Kovács and Sehgal [4] and Lee Theorem 3.3 and Corollary 3.1), it is evident the link between Lie identities satisfied by $\mathbb{F}G^+$ and group identities satisfied by $\mathcal{U}^+(\mathbb{F}G)$. In this setting, Jespers and Ruiz Marín [28, Theorem 4.1] obtained the following answer with respect to the commutativity of the symmetric units.

Theorem 3.6. *Let G be a torsion group and let R be a G-favorable integral domain. Then $\mathcal{U}^+(\mathbb{F}G)$ is an abelian group if and only if $\mathbb{F}G^+$ is a commutative ring.*

Lee, Sehgal and Spinelli [34] using the Theorem 3.5 confirm this link, finding necessary and sufficient conditions so that $\mathcal{U}^+(\mathbb{F}G)$ is nilpotent by proving the following.

Theorem 3.7. *Let $\mathbb{F}$ be an infinite field of characteristic different from 2, G a torsion group with an involution $*$ and let $\mathbb{F}G$ have the induced involution. Then $\mathcal{U}^+(\mathbb{F}G)$ is nilpotent if and only if $\mathbb{F}G^+$ is Lie nilpotent.*

3.3 Oriented Group Involutions

Holguín-Villa [27] extended the results obtained by Dooms and Ruiz [12] to the case of the oriented group involution $\circledast$ given by the expression (1).

Recall that a ring R with identity is said to be (von Neumann) regular if for any $x \in R$ there exists an $y \in R$ such that $xyx = x$. Villamayor [42] showed that the group algebra $\mathbb{F}G$ is regular if and only if G is locally finite and has no elements of order p. Note that in this case $\mathbb{F}G$ is semiprime.

In [27, Theorem 3.1] Holguín-Villa classified the groups with a regular group algebra over an infinite field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ for which the symmetric units satisfy a GI. The characterization is the following.

Theorem 3.8. *Let $\mathbb{F}$ be an infinite field with $\mathrm{char}(\mathbb{F}) \neq 2$ and let G be a non-abelian group such that $\mathbb{F}G$ is regular. Let $\sigma : G \to \{\pm 1\}$ be a nontrivial orientation and an involution $*$ on G. Suppose one of the following conditions holds:*

1. *$\mathbb{F}$ is uncountable,*
2. *All finite non-abelian subgroups of G which are $*$-invariant have no simple components in their group algebra over $\mathbb{F}$ that are non-commutative division algebras other than quaternion algebras.*

Then $\mathcal{U}^+(\mathbb{F}G) \in \mathrm{GI}$ if and only if one of the following conditions holds:

(i) $N = ker(\sigma)$ is an abelian group and $(G \setminus N) \subset G^+$;

(ii) *G and N have the* LC*-property, and there exists a unique nontrivial commutator s such that the involution $*$ is given by*

$$g^* = \begin{cases} g, & \text{if } g \in N \cap \zeta(G) \text{ or } g \in (G \setminus N) \setminus \zeta(G); \\ sg, & \text{otherwise.} \end{cases}$$

Consequently, $\mathcal{U}^+(\mathbb{F}G) \in \mathrm{GI}$ *if and only if* $\mathcal{U}^+(\mathbb{F}G)$ *is an abelian group.*

To handle group algebras which are not necessarily regular [27, Theorem 3.2], we need some extra lemmas, [27, Lemmas 3.5 and 3.6].

Finally in the case when the prime radical $\eta(\mathbb{F}G)$ of $\mathbb{F}G$ is nilpotent we characterize the groups for which the symmetric units $\mathcal{U}^+(\mathbb{F}G)$ do satisfy a group identity.

Theorem 3.9. *Let $g \mapsto g^*$ be an involution on a locally finite group G, $\sigma : G \to \{\pm 1\}$ a nontrivial orientation and $\mathbb{F}$ an infinite field with $\mathrm{char}(\mathbb{F}) = p \neq 2$. Suppose that the prime radical $\eta(\mathbb{F}G)$ of $\mathbb{F}G$ is a nilpotent ideal and that one of the following conditions holds:*

1. *$\mathbb{F}$ is uncountable,*
2. *All finite non-abelian subgroups of G/P which are $*$-invariant have no simple components in their group algebra over $\mathbb{F}$ that are non-commutative division algebras other than quaternion algebras.*

Then $\mathcal{U}^+(\mathbb{F}G) \in \mathrm{GI}$ *if and only if P is a finite normal subgroup and G/P is abelian or G/P and N/P are as in the Theorem 3.8.*

Suppose R is a G-favorable integral domain and $g \in G$ of finite order $\mathfrak{o}(g)$. Then, there exists $\alpha_g \in R$ such that $(1 - \alpha_g^{\mathfrak{o}(g)})$ is invertible in R. Jespers and Ruiz Marín [28] showed that $(g - \alpha_g)(g^* - \alpha_g)$ is a symmetric unit in the context of the group involution. This idea was generalized by Broche and Polcino Milies in [7, Theorem 3.1] for oriented involutions. Moreover, in the same theorem they showed the following.

Theorem 3.10. *Let G be a torsion group, R be a G-favorable integral domain and $\circledast$ an oriented group involution. Then, $\mathcal{U}^+(RG)$ is commutative if and only if RG^+ is commutative.*

Remark 3. *There are other Lie identities on symmetric elements that allow us to discuss the corresponding group identities on the symmetric units. We do not review these works in the present survey. The results show how, in some sense, polynomial identities satisfied by $\mathbb{F}G^+$ reflect group identities satisfied by $\mathcal{U}^+(\mathbb{F}G)$ and the latter ones can be lifted to the whole unit group of $\mathbb{F}G$. For more details we refer the reader to [41, theorems 31, 32 and 33] and [32].*

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