Lie properties of symmetric elements in regard to an oriented group involution

Alexander Holguín Villa*†

Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo-SP, Brazil

Supported by **CAPES** - Brazil and **CCP**, IME-USP, São Paulo, Brazil.

IME - Instituto de Matemática e Estatística

Introduction

Let $\mathbb{F}G$ denote the group algebra of the group G over the field $\mathbb{F}$ with $char(\mathbb{F}) \neq 2$. Any involution $*: G \to G$ can be extended $\mathbb{F}$-linearly to an algebra involution $*: \mathbb{F}G \to \mathbb{F}G$. A natural involution on G is the so-called classical involution, which maps $g \in G$ to g^{-1}.

Let $\sigma : G \to \{\pm 1\}$ be a homomorphism. If $*: G \to G$ is a group involution, an oriented group involution of $\mathbb{F}G$ is defined by

$$\alpha = \sum_{g \in G} \alpha_g g \mapsto \alpha^\dagger = \sum_{g\in G} \alpha_g \sigma(g) g^*.$$

We denote $\mathbb{F}G^+ = \{\alpha \in \mathbb{F}G : \alpha^\dagger = \alpha\}$ and $\mathbb{F}G^- = \{\alpha \in \mathbb{F}G : \alpha^\dagger = -\alpha\}$ the set of symmetric and skew-symmetric elements of $\mathbb{F}G$ under $\dagger$, respectively. In this poster we present some results about group algebras such that either $\mathbb{F}G^+$ or $\mathbb{F}G^-$ are Lie nilpotent (Lie n-Engel).

In the case that the involution on G is the classical involution, $g \mapsto g^{-1}$, the map $\dagger$ is precisely the oriented involution introduced by S. P. Novikov (1970) in the context of K-theory (see [CP11]).

In an associative ring R, we define the Lie product via $[x_1, x_2] = x_1x_2 - x_2x_1$ and, we can extended this recursively via

$$[x_1, ..., x_n, x_{n+1}] = [[x_1, ..., x_n], x_{n+1}].$$

Let S be a subset of R. We say that S is Lie nilpotent if there exists an $n \geq 2$ such that $[a_1, ..., a_n] = 0$ for all $a_i \in S$. The smallest such n is called the nilpotency index of S. For a positive integer n, we say that S is Lie n-Engel if

$$[a, \underbrace{b, ..., b}_{n \text{ times}}] = 0$$

for all $a, b \in S$. Obviously if S is Lie nilpotent then it is Lie n-Engel for some n.

Lie nilpotent (Lie n-Engel) group algebras have been the subject of a good deal of attention; indeed, it is interesting to know the extent to which the Lie properties of the symmetric (or skew-symmetric) elements determine the Lie properties of the whole group algebra.

- Began with Giambruno and Sehgal, in [GS93], under classical involution.
- G. Lee (see [L10, Section 3.3]).
- G. Lee also advanced in the knowledge of the Lie n-Engel property in $\mathbb{F}G^+$, [L10, Sections 3.1 and 3.2].
- Giambruno, Polcino Milies and Sehgal [GPS09], studied Lie properties in $\mathbb{F}G^+$, under group involution.
- Lee, Sehgal and Spinelli [LSS09], completed the last work.
- Recently, Castillo and Polcino Milies [CP11] have studied the Lie nilpotence and the Lie n-Engel properties in $\mathbb{F}G^+$ and $\mathbb{F}G^-$, under oriented classical involution.

In this poster we present some results about group algebras such that either $\mathbb{F}G^+$ or $\mathbb{F}G^-$ are Lie nilpotent (Lie n-Engel) under oriented group involutions; in particular we study the Lie nilpotence of $\mathbb{F}G^+$ in the case when $\tilde{\zeta}(G) = \{z^{-1}z^* : z \in \zeta(G)\}$ is an infinite set and, in this case, Lie nilpotence either $\mathbb{F}G^+$ or $\mathbb{F}G^-$, is equivalent to the Lie nilpotence of $\mathbb{F}G$.

Preliminaries

Let $\mathbb{F}$ be a field and let G be a group with a nontrivial homomorphism $\sigma : G \to \{\pm 1\}$ and an involution $*: G \to G$. Since σ is nontrivial, $char(\mathbb{F})$ must be different from 2.

If N denotes the kernel of σ, then N is a subgroup in G of $[G : N] = 2$. It is clear that the involution $\dagger$ coincides on the subalgebra $\mathbb{F}N$ with the algebra involution $*$. Also, we have that the symmetric elements in G, under $\dagger$, are the symmetric elements in N under $*$. If we denote the set of symmetric elements in G, under $*$, by G^+, then we can write $N^+ = N \cap G^+$. It is easy to see that, as an $\mathbb{F}$-module, $\mathbb{F}G^+$ is generated by the set

$$\mathcal{S} = N^+ \cup \{g + g^* : g \in N \setminus N^+\} \cup \{g - g^* : g \in G \setminus N, g^* \neq g\}$$

and $\mathbb{F}G^-$ is generated by

$$\mathcal{L} = (G \setminus N) \cap G^+ \cup \{g + g^* : g \in G \setminus N, g^* \neq g\} \cup \{g - g^* : g \in N, g^* \neq g\}.$$

In general, to classify Lie nilpotent group algebras $\mathbb{F}G$, we use **PI**-theory. Recall that a $\mathbb{F}$-algebra A satisfies a polynomial identity (we say that A is PI or $A \in$ **PI** for short) if there exists a nonzero polynomial $f(x_1, x_2, .., x_n)$ in the free associative $\mathbb{F}$-algebra on noncommuting variables $x_1, x_2, ..., x_n$ such that $f(a_1, a_2, ..., a_n) = 0$ for all $\{a_i\}_{i=1}^n \subseteq A$. Therefore Lie nilpotent (Lie n-Engel) algebras $\mathbb{F}G$ are **PI** group algebras satisfying a special identity. Group algebras satisfying a PI were classified by Passman and Isaacs-Passman, (see [P77, Corollaries 3.8 and 3.10]).

For a given prime p, an element $x \in G$ will be a called a p-element if its order is a power of p and it is called p'-element if its order is finite and, not divisible by p. The multiplicative commutator $g^{-1}h^{-1}gh$ of $g, h \in G$ is denoted by (g, h).

We begin this section with a lemma for **oriented group involutions**, which appears in Giambruno and Sehgal [GS93] for the classical involution, then in Giambruno, Polcino Milies and Sehgal [GPS09] for group involutions.

Lemma 1. *Let R be a semiprime ring with involution $*$ such that $2R = R$. If R^- (respectively R^+) is Lie n-Engel, then, $[R^-, R^-] = 0$ (respectively $[R^+, R^+] = 0$) and R satisfies St_4 the standard identity in four noncommuting variables, i.e., R satisfies*

$$St_4(x_1, x_2, x_3, x_4) = \sum_{\rho \in \mathcal{S}_4} (sgn\rho) x_{\rho(1)} x_{\rho(2)} x_{\rho(3)} x_{\rho(4)}.$$

We shall assume for the rest of the subsection that $\dagger$ is the classical oriented involution, i.e.,

$$\alpha = \sum_{g\in G} \alpha_g g \mapsto \alpha^\dagger = \sum_{g \in G} \alpha_g \sigma(g) g^{-1}.$$

Lemma 2. *If $\mathbb{F}G^-$ is Lie n-Engel, for some n, then every element of order 2 in N is central.*

We make use of the following technical lemma to obtain (Corollary 1), a known result, in a simpler way.

Lemma 3. *Let g and h be elements of G with $g^2 \neq 1$ and $h^2 \neq 1$. The following properties hold.*

(i) If $[g + g^{-1}, h + h^{-1}] = 0$, then either $gh = hg$ or $(g^\epsilon h^\eta)^2 = 1$ for all $\epsilon, \eta \in \{-1, 1\}$.

(ii) If $[g - g^{-1}, h - h^{-1}] = 0$, then either $gh = hg$ or $(g^\epsilon h^\eta)^2 = 1$ for all $\epsilon, \eta \in \{-1, 1\}$.

(iii) If $[g - g^{-1}, h + h^{-1}] = 0$, then either $gh \in \{hg, h^{-1}g\}$ or $o(g) = 4 = o(h)$ and $g^2 = h^2$.

Corollary 1. *Assume that $char(\mathbb{F}) = p > 2$ and that $\mathbb{F}G^-$ is Lie p^m-Engel (respectively $\mathbb{F}G^+$) for some $m \geq 1$. Let g and h be elements of G such that $g^2 \neq 1 \neq h^{2p^m}$. If $\sigma(g) = \sigma(h) = 1$ (respectively $\sigma(g) = \sigma(h) = -1$) then $(g, h^{p^m}) = 1$.*

Lie nilpotence when $|\tilde{\zeta}(G)| = \infty$

Let $\zeta = \zeta(G)$ denote the center of group G. In [GS93] it was proved that if $*$ is the classical involution and ζ^2 is infinite, and if $\mathbb{F}G^+$ or $\mathbb{F}G^-$ is Lie nilpotent of index n, then also $\mathbb{F}G$ is Lie nilpotent of index n. Also in [CP11] Castillo and Polcino Milies obtened a similar result for oriented classical involutions (Proposition 2.1). We adapted the proof of these results, to our situation.

Lemma 4. *Let G be a group such that $\tilde{\zeta}(G) = \{z^{-1}z^* : z \in \zeta(G)\}$ is infinite. If $\alpha \in \mathbb{F}G$ is such that $(\sigma(z)z^{-1}z^* - 1)\alpha = 0$, for all $z \in \zeta$, then $\alpha = 0$.*

Let now $\mathbb{F}\{x_1, x_1^*, ..., x_n, x_n^*, ...\}$ be the free algebra with involution and $\mathcal{R}$ an $\mathbb{F}$-algebra. Then $0 \neq f(x_1, x_1^*, ..., x_n, x_n^*) \in \mathbb{F}\{x_1, x_1^*, ..., x_n, x_n^*, ...\}$ is called a $*$-polynomial identity for $\mathcal{R}$ if $f(a_1, a_1^*, ..., a_n, a_n^*) = 0$ for all $\{a_i\}_{i=1}^n \subseteq \mathcal{R}$.

Theorem 1. *Let G be a group such that $\tilde{\zeta}$ is infinite. If $\mathbb{F}G$ satisfies a $\dagger$-polynomial identity ($\dagger$ − P.I) of degree n, then $\mathbb{F}G$ satisfies a polynomial identity of degree less than or equal to n.*

Remark 1. *The proof of this theorem shows that if f is a multilinear $\dagger$-P.I of degree n, then $(\sigma(z_i)z_i^{-1}z_i^* - 1)^n g$ vanishes in $\mathbb{F}G$ for all $z \in \zeta$ and g is the sum of all monomials of f containing no $*$.*

Corollary 2. *Let G be a group such that $\tilde{\zeta}$ is infinite. Then, $\mathbb{F}G^-$ or $\mathbb{F}G^+$ is Lie nilpotent of index n if and only if $\mathbb{F}G$ is Lie nilpotent of index n and, so G is nilpotent and p-abelian.*

Let σ be a nontrivial orientation from G onto $\mathcal{U}(\mathbb{F})$, where $\mathcal{U}(\mathbb{F}) = \mathbb{F}^\times$ is the group of units of the field $\mathbb{F}$ and, we consider the oriented group involution associated, $\alpha^\dagger = \Sigma\alpha_g\sigma(g)g^*$, then we have the following generalization of the last result:

Proposition 1. *Let $\mathbb{F}G$ denote the group algebra of the group G over the field $\mathbb{F}$ with $char(\mathbb{F}) \neq 2$ (or equivalently, there exists at least $g \in G$, such that $\sigma(g) = -1$), and let $\dagger : \mathbb{F}G \to \mathbb{F}G$ denote the involution defined by $\alpha = \Sigma\alpha_g g \mapsto \alpha^\dagger = \Sigma\alpha_g\sigma(g)g^*$, where $\sigma : G \to \mathcal{U}(\mathbb{F})$ is a group homomorphism and $*$ is an involution of the group G. Then, $\mathbb{F}G^+$ or $\mathbb{F}G^-$ is Lie nilpotent of index n if and only if $\mathbb{F}G$ is Lie nilpotent of index n.*

Groups without elements of order 2

Let G be a group without elements of order 2. Recall that a group G is said to be p-abelian if G', the commutator subgroup of G, is finite p-group, and 0-abelian will be taken to mean abelian.

It what follows, for a normal subgroup N of G we denote by $\Delta(G, N)$ the kernel of the natural map $\mathbb{F}G \xrightarrow{\Psi} \mathbb{F}(G/N)$ defined by

$$\sum_{g \in G} \alpha_g g \longmapsto \sum_{g\in G} \alpha_g gN$$

and $\Delta(G, G) = \Delta(G)$ is the augmentation ideal.

Lemma 5. *Let G be a group without elements of order 2 and $char(\mathbb{F}) = p$. Assume that $\mathbb{F}G^+$ or $\mathbb{F}G^-$ be Lie nilpotent. If the center of G has a nontrivial p'-element, then G is p-abelian.*

In [GPS09, Lemma 2.8] it was proved that if $\sigma \equiv 1$ and $*$ is any involution on the group G and, if $\mathbb{F}G^+$ ($char(\mathbb{F}) > 2$) is Lie n-Engel for some n, then for every symmetric element $g \in G$, g^{p^m} is central, for some $m > 0$. Currently, Castillo, Holguín and Polcino Milies are investigating (in preparation) which properties of Lie known for any involution $*$ defined on a group G and for the oriented classical involution, may be generalized to oriented group involutions, with some partial results already obtained; for instance, the oriented version of the lemma above mentioned, i.e., if $char(\mathbb{F}) > 2$ and $\mathbb{F}G^+$ is Lie n-Engel respect to oriented group involution $\dagger$ and, if $g \in N^+$, then g^{p^m} is central for some m.

In [Her76] Herstein studied a special class of rings with involution, called **semi-normal rings**.

Definition 1. *A ring R with involution $*$ is said to be semi-normal if $rr^* = 0$ implies $r^*r = 0$, for all $r \in R$.*

We clearly have two immediate classes of semi-normal rings:

- $rr^* = 0$ only if $x = 0$, in this case $*$ is called **positive definite** on R.
- $rr^* = r^*r$ for all $r \in R$. Such a ring we call **normal**.

In the following result we establish necessary and sufficient conditions on G and $N = ker(\sigma)$ under which the group algebra $\mathbb{F}G$ is normal, i.e., such that the $\dagger$-identity $\alpha\alpha^\dagger = \alpha^\dagger\alpha$ is satisfied.

Proposition 2. *Let $g \mapsto g^*$ denote an involution on a group G and let $\sigma : G \to \{\pm 1\}$ be a nontrivial homomorphism with $N = ker(\sigma)$. Let $\mathbb{F}G$ denote the group algebra of the group G over a commutative ring $\mathbb{F}$ with unity. Then, $\mathbb{F}G$ is normal if and only if one of the following conditions holds:*

(i) G is abelian;

(ii) $N = ker(\sigma)$ is abelian, $[G : N] = 2$ and we have that $x^* = x$ for $x \in G \setminus N$, $n^* = a^{-1}na$ for all $n \in N$ and for all $a \in G \setminus N$;

(iii) Both N and G are SLC-groups with canonical involution.

Theorem 2. *Let $\mathbb{F}$ be a field of $char(\mathbb{F}) \neq 2$ and let G be a group without 2-elements such that $\mathbb{F}G$ is semiprime. If $\mathbb{F}G$ is Lie n-Engel for some n. Then, G is abelian or $N = ker(\sigma)$ is abelian and $(G \setminus N) \subseteq G^+$. Moreover, $\mathbb{F}G$ is a normal group algebra.*

Remark 2. *Let G be a finite group of odd order. It is easy to see that the unique orientation σ that we can define on G is the trivial. In fact, for $g \in G$ $g^{|G|} = 1$, so $\sigma(g)^{|G|} = \sigma(g^{|G|}) = \sigma(1) = 1$. Since G is odd, we get that $\sigma(g) = 1$. Therefore, if $char(\mathbb{F}) = p > 2$ and $\mathbb{F}G^+$ is Lie n-Engel, then $\mathbb{F}G$ is Lie nilpotent.*

Acknowledgements

Some results if this Poster are part of the author's Ph.D. thesis [Hol12], to present to the Universidade de São Paulo (in preparation). The author thanks his advisor Professor César Polcino Milies, who introduced him to this fascinating topic and of course for his timely suggestions. He is also greatly indebted to his colleague and great friend John H. Castillo Gómez for helpful suggestions and unconditional friendship.

References

[BP06] O. Broche Cristo, C. Polcino Milies, Symmetric elements under oriented involutions in group rings, Comm. Algebra **34** (2006), 3347-3356.

[CP11] J. H. Castillo Gómez, C. Polcino Milies, Lie properties of symmetric elements under oriented involutions. To appear in Commun. Algebra (2012).

[GPS09] A. Giambruno, C. Polcino Milies, S. K. Sehgal, Lie properties of symmetric elements in group rings, J. Algebra **321** (2009), 890-902.

[GS89] A. Giambruno, S. K. Sehgal, A Lie property in group rings, Proc. Amer. Math. Soc. **105** (1989), 287-291.

[GS93] A. Giambruno, S. K. Sehgal, Lie nilpotency in group rings, Comm. Algebra **21** (1993), 4253-4261.

[Her76] I. N. Herstein, Rings with involution (1976). Univ. of Chicago Press, Chicago.

[Hol12] A. Holguín Villa, $*$-Identidades em Álgebras de Grupo (2012), Ph.D. thesis (in preparation), Universidade de São Paulo, São Paulo, Brazil.

[LSS09] G. T. Lee, S. K. Sehgal, E. Spinelli Lie nilpotency in group rings II, J. Pure Appl. Algebra **213** (2009), 1173-1178.

[L10] G. T. Lee, Group identities on units and symmetric units of group rings, Algebra and Applications, vol. 12, Springer-Verlag, London, 2010.

[PPS73] I. B. S. Passi, D. S. Passman, S. K. Sehgal, Lie solvable group rings, Canad. J.Math. **25** (1973), 748-757.

[P77] D. S. Passman, The algebraic structure of group rings, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, 1977.

[PS02] C. Polcino Milies, S. K. Sehgal, An introduction to group rings (2002). Dordrecht: Kluwer Academic Publishers.

*aholguin@ime.usp.br

†**CCP**: Comissão Coordenadora de Programa, IME-USP, São Paulo, Brazil.