Normal group algebras and oriented group involutions

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Introduction

Let RG denote the group algebra of the group G over a commutative ring R with unity. The group ring RG has a natural involution given by $\alpha = \Sigma\alpha_g g \mapsto \alpha^* = \Sigma\alpha_g g^{-1}$. This involution, known as the *classical involution*, appears as a technical tool to obtain results on units in a paper of G. Higman [Hig40]. In particular, it is used there for prove that if G is a finite abelian group, then $\mathbb{Z}G$ has non-trivial units unless either the orders of the elements of G divide four, or six, in which case $\mathbb{Z}G$ has only trivial units.

Given both a nontrivial homomorphism $\sigma : G \to \{\pm 1\}$ (called an orientation) and an involution $* : G \to G$ extended linearly to the group algebra RG, an oriented involution of RG is defined by

$$\alpha = \sum_{g\in G} \alpha_g g \mapsto \alpha^\dagger = \sum_{g\in G} \alpha_g \sigma(g) g^*.$$

Notice that, as σ is nontrivial, $char(R)$ must be different from 2. It is clear that, $\alpha \mapsto \alpha^\dagger$ is an involution of RG if and only if $gg^* \in N = ker(\sigma) = \{g \in G : \sigma(g) = 1\}$ for all $g \in G$. RG is said to be *normal* if and only if $\alpha\alpha^\dagger = \alpha^\dagger\alpha$, for all $\alpha \in RG$.

In the case that the involution on G is the classical involution, $g \mapsto g^{-1}$, the map $\dagger$ is precisely the oriented involution introduced by S. P. Novikov (1970) in the context of K-theory (see [CP11]).

Let now $R\{x_1, x_1^*, ..., x_n, x_n^*, ...\}$ be the free associative algebra with involution and A an R-algebra. Then $0 \neq f(x_1, x_1^*, ..., x_n, x_n^*) \in R\{x_1, x_1^*, ..., x_n, x_n^*, ...\}$ is called a $*$-polynomial identity for A if $f(a_1, a_1^*, ..., a_n, a_n^*) = 0$ for all $\{a_i\}_{i=1}^n \subseteq A$. Equivalently, the R-algebra A is called a **PI**-algebra. For instance, any commutative algebra would satisfy $x_1x_2 - x_2x_1 = 0$. The expression $[x_1, x_2] = x_1x_2 - x_2x_1$ is called the commutator of x_1 and x_2.

We denote $A^+ = \{\alpha \in A : \alpha^* = \alpha\}$ and $A^- = \{\alpha \in A : \alpha^* = -\alpha\}$ the set of symmetric and skew-symmetric elements of A under $*$, respectively. A question of general interest is to determine the extent to which the properties of A^+ or A^- determine the properties of the whole algebra A. One of the most famous and lovely results in this direction is the following theorem due to Amitsur (see [Her76, Theorem 6.5.2]):

Theorem 1. *Let R be a commutative ring with identity and A an R-algebra with involution $*$. If A satisfies a $*$-polynomial identity, then A satisfies a polynomial identity. In particular, if A^+ or A^- is* **PI**, *then A is* **PI**.

Of course, the polynomial identity which is satisfied by the R-algebra A is not necessarily the same as the one which is satisfied by the symmetric (skew-symmetric) elements.

Notice that normal group algebras RG are **PI** group algebras satisfying the special $\dagger$-identity $\alpha\alpha^\dagger = \alpha^\dagger\alpha$. Group algebras satisfying a PI were classified by Passman and Isaacs-Passman, (see [P77, Corollaries 3.8 and 3.10]).

Let $\zeta(G) = \zeta$ denote the center of the group G and recall that G is called *LC*-group if it is nonabelian and for every pair of elements $g, h \in G$, we have that $gh = hg$ if and only if either $g \in \zeta$, or $h \in \zeta$, or $gh \in \zeta$. A group is *SLC* if it is LC and has a unique nontrivial commutator.

Using unpublished results of Felzenszwab, Giambruno, Leal and Polcino, under *group involution*, we characterize group algebras RG which are normal in regard to an oriented group involution. The results depend on whether N is either abelian or an SLC-group.

Preliminaries

Let R be a commutative ring with unity and let G be a group with a nontrivial homomorphism $\sigma : G \to \{\pm 1\}$ and an involution $* : G \to G$. If N denotes the kernel of σ, then N is a subgroup in G of $[G : N] = 2$. It is clear that the involution $\dagger$ coincides on the subring RN with the ring involution $*$. Also, we have that the symmetric elements in G, under $\dagger$, are the symmetric elements in N under $*$. If we denote the set of symmetric elements in G, under $*$, by G^+, then we can write $N^+ = N \cap G^+$.

The groups G with the LC-property ("limited commutativity") were introduced by Goodaire and, have been described in Goodaire et al., [GJP96, Theorem III.3.3]. Moreover, by [GJP96, Proposition III.3.6] such groups are precisely those noncommutative groups with $G/\zeta(G) \cong C_2 \times C_2$, where C_2 is the cyclic group of order 2. Now, if G is *SLC*-group endowed with an involution $*$, then it has a unique nonidentity commutator s and the involution $*$ is defined by

$$g^* = \begin{cases} g & \text{if } g \text{ is central} \\ sg & \text{otherwise} \end{cases} \qquad (1)$$

and we refer to this as the *canonical* involution on an SLC-group.

The additive commutator $\alpha\beta - \beta\alpha$, for $\alpha, \beta \in RG$, will be denoted by the Lie bracket $[\alpha, \beta]$ and the multiplicative commutator $g^{-1}h^{-1}gh$ of $g, h \in G$ will be denoted by (g, h).

Remark 1. *If G is a group with the LC-property, then for all $g \in G$ $g^2 = gg$ is central. Thus, since $(g, h) = g^{-1}h^{-1}gh = g^{-2}gh^{-1}gh^{-1}h^2 = g^{-2}(gh^{-1})^2h^2$, commutators are central in a LC-group G.*

Now, suppose that RG is a *normal* ring and let $N = ker(\sigma)$. Then RN is also *normal*, and thus, by [FGLM10], N is an abelian group or N is an *SLC*-group with canonical involution.

Some lemmas

We begin with some lemmas, which are the extended version of those established in [FGLM10].

Lemma 1. *Suppose that RG is normal and let $g, h \in G$, then:*
(i) If $\sigma(g)\sigma(h) = 1$, then either $gh = hg$ or $gh = g^*h^*$.
(ii) If $\sigma(g)\sigma(h) = -1$, then either $gh = hg$ or $gh = (gh)^*$.

Lemma 2. *Suppose that RG is normal and let $g \in G$. Then one of the following conditions holds:*
(i) If either $\sigma(g) = \sigma(h) = 1$ or if $\sigma(g) = -1$ and $\sigma(h) = 1$, then $g^2h = hg^2$.
(ii) If either $\sigma(g) = 1$ and $\sigma(h) = -1$ or sif $\sigma(g) = \sigma(h) = -1$, then $g^2h = (g^2h)^*$.

In particular, for $n, m \in N$, $(n^2, m) = 1$.

Lemma 3. *If RG is a normal group algebra, then*

$$N^+ = N \cap G^+ \subseteq \zeta(G).$$

In particular, for all $g \in G$, $gg^* = g^*g$.

Lemma 3. *Let $g, h \in G$ such that $(g, h) \neq 1$ and RG a normal group algebra. Then one of the following conditions holds:*
(i) $\sigma(g) = \sigma(h) = 1$, $g^* = (g, h)g$, $h^* = (g, h)h$, $\gamma_2(\langle g, h\rangle)$ has order 2;
(ii) $\sigma(g) = -1$ and $\sigma(h) = 1$, $g^* = g$, $h^* = (g, h)h$, $(g^2, h) = 1$, $(gh)^2 = (hg)^2$, $\gamma_2(\langle g, h\rangle)$ has order 2;
(iii) $\sigma(g) = 1$ and $\sigma(h) = -1$, $g^* = (g, h)g$, $h^* = h$, $(h^2, g) = 1$, $(gh)^2 = (hg)^2$;
(iv) $\sigma(g) = \sigma(h) = -1$, $g^* = g$, $h^* = h$, $(g^2, h) = 1$, $(h^2, g) = 1$, $(gh)^2 = (hg)^2$.

Main results

In [Her76] Herstein studied a *special class* of rings with involution, called **semi-normal rings**.

Definition 1. *A ring R with involution $*$ is said to be* semi-normal *if $rr^* = 0$ implies $r^*r = 0$, for all $r \in R$.*

Clearly normal rings are semi-normal. The involution $*$ of the ring R is called **positive definite** if $r = 0$ implies $rr^* = 0$.

Other celebrated theorem due to Amitsur [Her76, Theorem 6.5.1] and that extends the Theorem 1, establishes a relationship between $*$-polynomial identities and the identities which does not include variables with $*$, satisfied for a ring R. More exactly we have:

Theorem 2. *If $f(x_1, x_1^*, ..., x_r, x_r^*)$ is a polynomial identity of degree d for the $\mathbb{F}$-algebra R, then R satisfies $St_{2d}(x_1, x_2, ..., x_{2d})^m$ for some m, the* standard identity *in $2d$ variables. If R is* semi-prime *then $m = 1$.*

If R is normal ring, by the last theorem, R satisfies $St_4(x_1, x_2, x_3, x_4)^m$ an if R is semi-prime then R satisfies $St_4(x_1, x_2, x_3, x_4)$, so is imbeddable in 2×2 matrices over a commutative ring.

This result, with $rr^* = r^*r$ for all $r \in R$, can be obtained also by completely elementary arguments.

In the following result we establish necessary and sufficient conditions on G and $N = ker(\sigma)$ under which the group algebra $\mathbb{F}G$ is *normal*, i.e., such that the $\dagger$-identity $\alpha\alpha^\dagger = \alpha^\dagger\alpha$ is satisfied.

Theorem 3. *Let $g \mapsto g^*$ denote an involution on a group G and let $\sigma : G \to \{\pm 1\}$ be a nontrivial homomorphism with $N = ker(\sigma)$. Let $\mathbb{F}G$ denote the group algebra of the group G over a commutative ring $\mathbb{F}$ with unity. Then, $\mathbb{F}G$ is normal if and only if one of the following conditions holds:*
(i) G is abelian;
(ii) $N = ker(\sigma)$ is abelian, $[G : N] = 2$ and we have that $x^* = x$ for $x \in G \setminus N$, $n^* = a^{-1}na$ for all $n \in N$ and for all $a \in G \setminus N$;
(iii) Both N and G are SLC-groups with canonical involution.

Examples (i) *Let $N = 2^n$ with $n \geq 1$ and let G be the group given by $G =< a, b : a^{2N} = b^2 = 1, ba = a^{N+1}b >$. Then, $G/\zeta(G) = G/ < a^2 > \cong C_2 \times C_2$. Thus, by [GJP96, Proposition III.3.6] G is an SLC-group.*

(ii) *Let G be the group presented as follows*

$$G = \langle x_1, x_2, x_3 : x_i^4 = (x_i^2, x_j) = ((x_i, x_j), x_k) = 1; i \neq j \neq k\rangle.$$

Then, $exp(G/\zeta(G)) = 2$ and $g, h \notin \zeta(G)$ are such that (g,h)=1, if and only if they lie in the same coset of the $\zeta(G)$. Therefore, G has the LC-property, but G has three nonidentity commutators (x_1, x_2), (x_1, x_3) and (x_2, x_3). Thus, the LC-property and the presence of a unique commutator $1 \neq s$ in a group G, are independent conditions.

(iii) *Let $\mathfrak{R}$ be a ring with an involution $*$ (in particular if $\mathfrak{R} = RG$) and, for $r \in \mathfrak{R}$, define respectively (see [GJP96]) the trace and norm of r by*

$$t(r) = r + r^* \text{ and } n(r) = rr^*.$$

If $\mathfrak{R}$ is a normal ring, then for all $r_1, r_2 \in \mathfrak{R}$, $t(r_1r_2) = t(r_2r_1)$, since

$$\begin{aligned} t(r_1r_2) &= n(r_1 + r_2^*) - n(r_1) - n(r_2^*) \\ &= n(r_1^* + r_2) - n(r_1^*) - n(r_2) \\ &= t(r_2r_1). \end{aligned} \qquad (2)$$

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