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"Funciones zeta locales de Igusa y sumas exponenciales de polinomios aritméticamente no degenerados"

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M. en C. Adriana Alexandra Albarracín Mantilla

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Directores de la Tesis:

Dr. Wilson Álvaro Zúñiga Galindo

Dr. Edwin León Cardenal

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"Igusa's Local Zeta Functions and Exponential Sums for Arithmetically Non Degenerate Polynomials"

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MSc. Adriana Alexandra Albarracín Mantilla

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Dr. Edwin León Cardenal

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Doctoral Dissertation Jury

Dr. Edwin León Cardenal

Centro de Investigación en Matemáticas

Unidad Zacatecas

Dr. Enrique Reyes Espinoza

Departamento de Matemáticas

CINVESTAV - Zacatenco

Dr. Iakov Mostovoi

Departamento de Matemáticas

CINVESTAV - Zacatenco

Dr. Pedro Luis Del Ángel

Centro de Investigación en Matemáticas

Guanajuato, Guanajuato

Dr. Wilson Álvaro Zúñiga Galindo

Departamento de Matemáticas

CINVESTAV - Querétaro

Reviewers

Dr. Felipe Zaldívar Cruz
Departamento de Matemáticas
Universidad Autónoma Metropolitana
Ciudad de México

Dr. Florian Luca
University of the Witwatersrand
School of Mathematics
South Africa

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Abstract

We study the twisted local zeta function associated to a polynomial in two variables with coefficients in a non–Archimedean local field of arbitrary characteristic. Under the hypothesis that the polynomial is arithmetically non degenerate, we obtain an explicit list of candidates for the poles in terms of geometric data obtained from a family of arithmetic Newton polygons attached to the polynomial. The notion of arithmetical non degeneracy due to Saia and Zúñiga-Galindo is weaker than the usual notion of non degeneracy due to Kouchnirenko. As an application we obtain asymptotic expansions for certain exponential sums attached to these polynomials.

Resumen

En esta disertación estudiamos la función zeta local torcida asociada a polinomios en dos variables con coeficientes en un campo local no Arquimediano de característica arbitraria. Bajo la hipótesis que el polinomio es aritméticamente no degenerado, obtenemos una lista explícita de candidatos a polos, en términos de los datos geométricos obtenidos de una familia de polígonos aritméticos de Newton asociada al polinomio. La noción de aritméticamente no degenerado en el sentido de Saia y Zúñiga-Galindo es más general que la noción usual de no degeneración de Kouchnirenko. Además como una aplicación obtenemos expansiones asintóticas para ciertas sumas exponenciales asociadas a estos polinomios.

Contents

Overview

The local zeta functions over local fields, i.e. $\mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{F}_p((T))$, are ubiquitous objects in mathematics and mathematical physics see e.g. [2, 5–7, 10, 12, 15, 17–19, 22, 26, 28, 30, 32–34, 36–38]. For instance these objects are deeply connected with string and Feynman amplitudes. Let us mention that the works of Speer [28] and Bollini, Giambiagi and González Domínguez [7] on regularization of Feynman amplitudes in quantum field theory are based on the work of Gel'fand and Shilov [17] on the analytic continuation of Archimedean local zeta functions. For connections with String theory see e.g. [9] and the references therein. In the non-Archimedean setting, for instance in the p-adic case, the local zeta functions are related to the number of polynomial congruences mod p^m and exponential sums mod p^m. There are many intriguing conjectures connecting the poles of non-Archimedean local zeta functions, with the topology of complex singularities, see e.g. [12–14, 16, 19, 25, 27, 30–32, 35, 37, 38].

Let K be a non–Archimedean local field of arbitrary characteristic with valuation v, let O_K be its ring of integers with group of units $O_K^{\times}$, let P_K be the maximal ideal in O_K. We fix a uniformizer parameter $\mathfrak{p}$ of O_K. We assume that the residue field of O_K is $\mathbb{F}_q$, the finite field with q elements. The absolute value for K is defined by $|z| := |z|_K = q^{-v(z)}$, and for $z \in K^{\times}$, we define the angular component of z by $ac(z) = z\mathfrak{p}^{-v(z)}$. We consider $f(x, y) \in O_K[x, y]$ a non-constant polynomial and χ a character of $O_K^{\times}$, that is, a continuous homomorphism from $O_K^{\times}$ to the unit circle, considered as a subgroup of $\mathbb{C}^{\times}$. When $\chi(z) = 1$ for any $z \in O_K^{\times}$, we will say that χ is the trivial character and it we denote it as χ_{triv}. We associate to these data the local zeta function,

$$Z(s, f, \chi) := \int_{O_K^2} \chi(ac\ f(x, y))\ |f(x, y)|^s\ |dxdy|, \quad s \in \mathbb{C},$$

where $Re(s) > 0$, and $|dxdy|$ denotes the Haar measure of $(K^2, +)$ normalized such that the measure of O_K^2 is one.

It is not difficult to see that $Z(s, f, \chi)$ is holomorphic on the half plane $Re(s) > 0$. Furthermore, in the case of characteristic zero, Igusa [20] and Denef [11] proved that $Z(s, f, \chi)$ is a rational function of q^{-s}, for an arbitrary polynomial in several variables. When $char(K) > 0$, new techniques are needed since there is no a general theorem of resolution of singularities, nor an equivalent method of p- adic cell decomposition. In [21] Igusa introduced the stationary phase formula (SPF) and conjectured that by using it, the rationality of the local zeta functions can be established in arbitrary characteristic. This conjecture has been verified in several cases, see e.g. [23, 27, 37] and the references therein.

A considerable advance in the study of local zeta functions in arbitrary characteristic has been obtained for a large class of polynomials which satisfy a non–degeneracy condition. Roughly speaking, the idea is to attach a Newton polyhedron to the polynomial f and then define a non degeneracy condition with respect to the Newton polyhedron. Then one may construct a toric variety associated to the Newton polyhedron, and use toric resolution of singularities in order to establish a meromorphic continuation of $Z(s, f, \chi)$, see e.g. [2, 25] for a good discussion about the Newton polyhedra technique in the study of local zeta functions. The first use of this approach was pioneered by Varchenko [29] in the Archimedean case. After Varchenko's article, several authors have been used his methods to study local zeta functions, oscillatory integrals, and exponential sums, see for instance [13, 14, 24, 25, 27, 32, 37] and the references therein.

In this dissertation we study local zeta functions for arithmetically non-degenerate polynomials. In [27] Saia and Zúñiga-Galindo introduced the notion of arithmetically non–degeneracy for polynomials in two variables, this notion is weaker than the classical notion of non–degeneracy due to Kouchnirenko, see e.g. [2]. They used this notion to study local zeta functions $Z(s, f, \chi_{triv})$ when f is an arithmetically non–degenerate polynomial with coefficients in a non–Archimedean local field of arbitrary characteristic. They established the existence of a meromorphic continuation for $Z(s, f, \chi_{triv})$ as a rational function of q^{-s}, and gave an explicit list of candidate poles for $Z(s, f, \chi_{triv})$ in terms of a family of arithmetic Newton polygons which are associated with f. In this dissertation, we extend the results of Saia and Zúñiga-Galindo to twisted local zeta function $Z(s, f, \chi)$, for χ arbitrary, and f a polynomial in two variables with coefficients in a local field of arbitrary characteristic which is non-degenerate in the sense of Saia and Zúñiga-Galindo.

By using the techniques of [27] we obtain an explicit list of candidate poles of $Z(s, f, \chi)$ in terms of the equations of the straight segments defining the boundaries of the arithmetic Newton polygon attached to f.

The following result describes the poles of the meromorphic continuation of $Z(s, f, \chi)$ for arbitrary χ:

Theorem 2.5.1 *Let $f(x, y) \in K[x, y]$ be a non-constant polynomial. If $f(x, y)$ is arithmetically non-degenerate with respect to its arithmetic Newton polygon $\Gamma^A(f)$, then the real parts of the poles of $Z(s, f, \chi)$ belong to the set*

$$\{-1\} \cup \mathcal{P}(\Gamma^{geom}(f)) \cup \mathcal{P}(\Gamma^A(f)).$$

In addition $Z(s, f, \chi)$ vanishes for almost all χ.

The main contribution of this dissertation is the study of the exponential sums mod $\mathfrak{p}^m$ attached to arithmetically non-degenerate polynomials. Exponential sums mod $\mathfrak{p}^m$ have been studied intensively, see e.g. [3, 4, 14, 16, 37].

By fixing an additive character $\Psi : K \to \mathbb{C}$, exponential sums mod $\mathfrak{p}^m$ can written as

$$E(z, f) = \int_{O_K^2} \Psi(zf(x, y)) \; |dx \; dy|,$$

where $z = \mathfrak{p}^m u, \; u \in O_K^\times$. A central problem consists in describing the asymptotic behavior of $E(z, f)$ as $|z| \to \infty$. Our main result about exponential sums mod $\mathfrak{p}^m$ for arithmetically non-degenerate polynomials is the following:

Theorem 3.1.1 *Let $f(x, y) \in K[x, y]$ be a non constant polynomial which is arithmetically modulo $\mathfrak{p}$ non–degenerate with respect to its arithmetic Newton polygon. Assume that $C_f \subset f^{-1}(0)$ and assume all the notation introduced previously. Then the* *following assertions hold.*

1. *For $|z|$ big enough, $E(z,f)$ is a finite linear combination of functions of the form*

$$\chi(ac\ z)|z|^{\lambda}(\log_q\ |z|)^{j_\lambda},$$

with coefficients independent of z, and $\lambda \in \mathbb{C}$ a pole of $Z(s,f,\chi)$ (with $\chi|_{1+\mathfrak{p}O_K} = \chi_{triv}$) or $(1-q^{-s-1})Z(s,f,\chi_{triv})$, where

$$j_\lambda = \begin{cases} 0 & \text{if } \lambda \text{ is a simple pole} \\ 0,1 & \text{if } \lambda \text{ is a double pole.} \end{cases}$$

Moreover all the poles λ appear effectively in this linear combination.

2. *Assume that $\beta := \max\{\beta_{\Gamma^{geom}}, \beta_{\Gamma_\theta^A}\} > -1$. Then for $|z| > 1$, there exist a positive constant $C(K)$, such that*

$$|E(z,f)| \leqslant C(K)|z|^{\beta}\log_q\ |z|.$$

The results presented in this dissertation will be published in an article written in collaboration with Professor Edwin León-Cardenal in the Journal de Théorie des Nombres de Bordeaux. I am very grateful to Professor Wilson A. Zúñiga-Galindo for suggesting me the thematic for this dissertation and for their kind guidance during the whole development of the present work.

This dissertation is organized as follows. In Chapter 1, we review some basic facts about local zeta functions and exponential sums $\mod \mathfrak{p}^m$. We also review Igusa's stationary phase formula, which will be used along this dissertation. In Chapter 2, we prove Theorem 2.5.1 and give some examples. The full calculation of these examples is very long, for this reason in Chapter 2 we only sketch a small part of them. In Chapter 3, we prove Theorem 3.1.1. In Appendices A and B we have included the full calculation of the examples sketched in Section 2.3 of Chapter 2.

Finally we want to mention that as a future project we would like to study the extension of the ideas in this document to the case of polynomials in an arbitrary number if variables. One of the main difficulties in this task is to find the “right” generalization of the arithmetic Newton polyhedron of a polynomial function f in more than two variables.

Chapter 1

Preliminaries

For the sake of completeness, we review some basic concepts about the theory of local zeta functions on non-Archimedean fields of arbitrary characteristic. We also make a brief presentation of Igusa's stationary phase formula as in [37], in section 1.2.2 we review the basic aspects of exponential sums mod $\mathfrak{p}^m$ defined over non–Archimedean local fields. Finally we present an explicit formula for $Z(s, f, \chi)$ for polynomials that are non-degenerate with respect to their Newton polyhedron, see sections 1.3, 1.3.1 and 1.3.2.

1.1 Local Zeta Functions

Let K be a non–Archimedean local field, which is a locally compact topological field with respect to a non-discrete topology. By a well-known theorem, see e.g. [33], a such field is isomorphic (as a topological field) to a finite extension of the field of p-adic numbers $\mathbb{Q}_p$, or isomorphic to a finite extension of $\mathbb{F}_p((T))$, the field of formal Laurent series with coefficients in a finite field $\mathbb{F}_p$. Let $|\cdot|_K := |\cdot|$ be the absolute value of K (K is a complete metric space for the distance induced by $|\cdot|$). Let O_K be the valuation ring of K which is

$$O_K = \{x \in K; |x| \leqslant 1\}.$$

Let P_K be the unique maximal ideal of O_K, which is a principal ideal. We fix a generator $\mathfrak{p}$ of P_K, which is also called a uniformizer parameter of O_K. The quotient field O_K/P_K

is called the residue field of K, and it is the finite field of cardinality $q = p^e$, p a prime number. The group of units of O_K is $O_K^\times = \{x \in O_K : |x| = 1\}$. We will assume that $|\cdot|$ is a normalized absolute value, which means that $|x| = q^{-v(x)}$, where $v(x) \in \mathbb{Z} \cup \{\infty\}$ is a valuation on K. The canonical mapping $O_K \to O_K/P_K \cong \mathbb{F}_q$ is called the reduction mod $\mathfrak{p}$. We denote by R_K a fixed set of representatives of $\mathbb{F}_q$ in O_K. Then every element x of $K \setminus \{0\}$ can be represented as a convergent series with respect to $|\cdot|$ as follows:

$$x = \mathfrak{p}^{m_0} \sum_{m=0}^{\infty} a_m \mathfrak{p}^m, \;\; a_m \in R_K, a_0 \neq 0,$$

where $m_0 = v(x)$.

Example 1.1.1. *The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers with respect to the $p-$ adic norm $|\cdot|_p$, which is defined as*

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-r} & \text{if } x = p^r \frac{a}{b}, \end{cases}$$

where a and b are integers co-prime with p.

The group $(K^n, +)$ is a locally compact group, where K^n is endowed with the product topology. We denote by $|dx| = |dx_1 \cdots dx_n|$ the Haar measure on $(K^n, +)$ normalized so that $\int_{O_K^n} |dx| = 1$.

A quasicharacter of $K^\times$ is a continuous homomorphism $\omega : K^\times \to \mathbb{C}^\times$. The set of quasicharacters, that we will denote by $\Omega(K^\times)$, has an Abelian group structure, and to a given complex number s we may associate a quasicharacter $\omega_s \in \Omega(K^\times)$ by setting $\omega_s(x) = |x|_K^s$. Once we pick $\omega(\mathfrak{p}) = q^{-s}$, for every $\omega \in \Omega(K^\times)$, one has

$$\omega(x) = \omega_s(x) \chi(ac\, x), \tag{1.1.1}$$

where $\chi := \omega \mid_{O_K^\times}$, is a group homomorphism with finite image. Put formally $\chi(0) = 0$. For $z \in K$, we define the angular component of z by $ac(z) = z\mathfrak{p}^{-v(z)}$. Equation (1.1.1) shows that

$$\Omega\left(K^{\times}\right) \simeq \mathbb{C}/\left(2\pi\sqrt{-1}/\ln q\right) \times \left(O_K^{\times}\right)^*,$$

where $\left(O_K^{\times}\right)^*$ is the group of characters of $O_K^{\times}$; therefore $\Omega\left(K^{\times}\right)$ is a one dimensional complex manifold. Note that $\sigma\left(\omega\right) := \mathrm{Re}(s)$ depends only on ω, and $\left|\omega\left(x\right)\right|_{\mathbb{C}} = \omega_{\sigma(\omega)}\left(x\right)$, thus it makes sense to define the following open subset of $\Omega\left(K^{\times}\right)$,

$$\Omega_{(a,b)}\left(K^{\times}\right) = \left\{\omega \in \Omega\left(K^{\times}\right); \sigma\left(\omega\right) \in (a,b) \subseteq \mathbb{R}\right\}.$$

Then the local zeta functions $Z(s, f, \chi)$ of f and χ is defined by the integral

$$Z(s, f, \chi) = \int\limits_{O_K^n} \chi(ac\ f(x))\ |f(x)|^s\ |dx|,$$

for $s \in \mathbb{C}$ satisfying $Re(s) > 0$. In the case in which χ is the trivial character we simply write $Z(s, f)$. The local zeta functions admit a meromorphic continuation to the complex plane as rational functions of q^{-s}, see [22, Theorem 8.2.1].

1.1.1 Poincaré Series

Let $f(x) \in O_K[x_1, \cdots, x_n]$ be a non-constant polynomial. A classical problem in number theory consists in studying the number of solutions of polynomial congruences $f(x) \equiv 0 \pmod{P_K^m}$, more precisely, to study the behavior of the numbers

$$N_m := \#\{x \in (O_K/P_K^m)^n; f(x) \equiv 0 \pmod{P_K^m}\},$$

with $N_0 = 1$, as m tends to infinity. To study this problem one introduces the Poincaré series

$$P(t) = \sum_{m \geqslant 0} N_m q^{-mn} t^m,\ t \in \mathbb{C},$$

with $|t| < 1$. The following formula established a relation between $P(t)$ and $Z(s, f)$

$$P(t) = \frac{1 - tZ(s, f)}{1 - t}, t = q^{-s},$$

when $Re(s) > 0$, see [22, Theorem 8.2.2]. This formula shows that the local zeta functions have arithmetical nature. In [8], Borevich and Shafaverich conjectured in the 60's, that in the case of characteristic zero, $P(t)$ is a rational function. This conjecture was established by Igusa in the middle of the 70's as a Corollary of the following Theorem:

Theorem 1.1.1 ([22, Theorem 8.2.1])**.** *Let K be a local field of characteristic zero. Let $f(x)$ be a non-constant polynomial in $K[[x_1, \cdots, x_n]]$. There exist a finite number of pairs $(N_E, \nu_E) \in (\mathbb{N} \setminus \{0\}) \times (\mathbb{N} \setminus \{0\}), E \in T$, such that*

$$\prod_{E \in T} (1 - q^{\nu_E - sN_E}) Z(s, f, \chi)$$

is a polynomial in q^{-s} with rational coefficients.

1.2 Some Technical Results

In this section, we summarized some results of [22], that will be used later on.

Lemma 1.2.1 ([22, Lemma 8.2.1])**.** *Take $a \in O_K$, χ a character of $O_K^{\times}$, $e \in \mathbb{N}$. Then*

$$\int_{a + \mathfrak{p}^e O_K} \chi(ac(x))^N |x|^{sN+n-1}\, dx$$

$$= \begin{cases} \frac{(1-q^{-1})q^{-en-eNs}}{1-q^{-n-Ns}} & \text{if } a \in \mathfrak{p}^e O_K, \chi^N = \chi_{triv} \\ q^{-e}\chi(ac(a))^N |a|^{sN+n-1} & \text{if } a \notin \mathfrak{p}^e O_K, \chi^N|_{1+\mathfrak{p}^e a^{-1} O_K} = \chi_{triv} \\ 0 & \text{all other cases.} \end{cases}$$

Proof. The proof of the lemma is an easy variation of the one given in [22]. □

The next result is an easy consequence of Lemma 1.2.1 and will be used frequently in the following sections.

Lemma 1.2.2. *Take $h(x,y) \in O_K[x,y]$, then*

$$\sum_{(\overline{x}_0,\overline{y}_0)\in({\mathbb{F}_q}^{\times})^2} \int_{O_K} \chi(ac\ (h(x_0,y_0)+\mathfrak{p}z))\ |h(x_0,y_0)+\mathfrak{p}z|^s\ |dz|$$

equals

$$\begin{cases} \frac{q^{-s}(1-q^{-1})N}{(1-q^{-1-s})} + (q-1)^2 - N & \text{if } \chi = \chi_{triv} \\ \sum\limits_{\substack{(\overline{x}_0,\overline{y}_0)\in({\mathbb{F}_q}^{\times})^2 \\ \overline{h}(\overline{x}_0,\overline{y}_0)\neq 0}} \chi(ac(h(x_0,y_0))) & \text{if } \chi \neq \chi_{triv} \text{ and } \chi|_U = \chi_{triv} \\ 0 & \text{all other cases,} \end{cases}$$

where $N = Card\{(\overline{x}_0,\overline{y}_0) \in ({\mathbb{F}_q}^{\times})^2 \mid \overline{h}(\overline{x}_0,\overline{y}_0) = 0\}$, and $U = 1+\mathfrak{p}O_K$.

Proof. We have that

$$\begin{aligned} &\sum_{(\overline{x}_0,\overline{y}_0)\in({\mathbb{F}_q}^{\times})^2} \int_{O_K} \chi(ac\ (h(x_0,y_0)+\mathfrak{p}z))\ |h(x_0,y_0)+\mathfrak{p}z|^s\ |dz| \\ &= \sum_{\substack{(\overline{x}_0,\overline{y}_0)\in({\mathbb{F}_q}^{\times})^2 \\ \overline{h}(\overline{x}_0,\overline{y}_0)=0}} \int_{O_K} \chi(ac\ (h(x_0,y_0)+\mathfrak{p}z))\ |h(x_0,y_0)+\mathfrak{p}z|^s\ |dz| \\ &+ \sum_{\substack{(\overline{x}_0,\overline{y}_0)\in({\mathbb{F}_q}^{\times})^2 \\ \overline{h}(\overline{x}_0,\overline{y}_0)\neq 0}} \int_{O_K} \chi(ac\ (h(x_0,y_0)+\mathfrak{p}z))\ |h(x_0,y_0)+\mathfrak{p}z|^s\ |dz|. \end{aligned} \tag{1.2.1}$$

By Lemma 1.2.1 the first sum in the right hand side of (1.2.1) is equal to

$$\int_{O_K} \chi\left(ac\ \left(\frac{h(x_0,y_0)}{\mathfrak{p}}+z\right)\right)|dz| = \sum_{(\overline{x}_0,\overline{y}_0)\in({\mathbb{F}_q}^{\times})^2} \int_{\frac{h(x_0,y_0)}{\mathfrak{p}}+O_K} \chi(ac\ (z)\ |dz|,$$

$$= \begin{cases} \frac{q^{-s}(1-q^{-1})N}{(1-q^{-1-s})} & \textit{if } \chi=\chi_{triv} \\ 0 & \textit{all other cases.}\end{cases}$$

Now, for the second sum in the right hand side of (1.2.1), we have

$$\sum_{\substack{(\overline{x}_0,\overline{y}_0)\in({\mathbb{F}_q}^{\times})^2\\ \overline{h}(\overline{x}_0,\overline{y}_0)\neq 0}} \int_{O_K} \chi(ac\ (h(x_0,y_0)+\mathfrak{p}z))\ |h(x_0,y_0)+\mathfrak{p}z|^s\ |dz|$$

$$= \sum_{\substack{(\overline{x}_0,\overline{y}_0)\in({\mathbb{F}_q}^{\times})^2\\ \overline{h}(\overline{x}_0,\overline{y}_0)\neq 0}} \int_{h(x_0,y_0)+\mathfrak{p}O_K} \chi(ac\ w)|dw|,$$

$$= \begin{cases} (q-1)^2-N & \textit{if } \chi=\chi_{triv} \\ \sum\limits_{\substack{(\overline{x}_0,\overline{y}_0)\in({\mathbb{F}_q}^{\times})^2\\ \overline{h}(\overline{x}_0,\overline{y}_0)\neq 0}} \chi(ac(h(\overline{x}_0,\overline{y}_0))) & \textit{if } \ \chi\neq\chi_{triv} \textit{ and } \ \chi|_U=\chi_{triv} \\ 0 & \textit{all other cases.}\end{cases}$$

where $N = \mathrm{Card}\{(\overline{x}_0,\overline{y}_0)\in({\mathbb{F}_q}^{\times})^2 \mid \overline{h}(\overline{x}_0,\overline{y}_0)=0\}$, and $U = 1+\mathfrak{p}O_K$. □

1.2.1 Igusa's stationary phase formula

There is an interactive procedure that allows in many cases to calculate the local zeta functions in an explicit way. We recall here the stationary phase formula. Let c_χ be the conductor of a character χ of $O_K^{\times n}$ is defined as the smallest $c\in\mathbb{N}\setminus\{0\}$ such that χ is trivial on $1+\mathfrak{p}^cO_K$.

Denote by $\bar{x}$ the reduction mod $\mathfrak{p}$ of $x\in O_K$, we denote by $\overline{f}(x)$ the reduction of the coefficients of $f(x)\in O_K[x]$ (we assume that not all of the coefficients of f are in

P_K). We fix a set of representatives R of $\mathbb{F}_q$ in O_K, that is, R^n is mapped bijectively onto $\mathbb{F}_q^n$ by the canonical homomorphism $O_K^n \to (O_K/P_K)^n \simeq \mathbb{F}_q^n$. Now take $\overline{T} \subseteq \mathbb{F}_q^n$ and denote by T its preimage under the aforementioned homomorphism, we denote by $S_T(f)$ the subset of R^n mapped bijectively to the set of singular points of $\overline{f}$ in $\overline{T}$. We define also

$$\nu_T(\bar{f},\chi) := \begin{cases} q^{-n}\mathrm{Card}\{\bar{t} \in \overline{T} \mid \bar{f}(\bar{t}) \neq 0\} & \textit{if } \chi = \chi_{triv} \\ \\ q^{-nc_\chi} \sum\limits_{\{t \in T \mid \bar{f}(\bar{t}) \neq 0\} \mod P^{c_\chi}} \chi(ac\ (f(t))) & \textit{if } \ \chi \neq \chi_{triv}, \end{cases}$$

and

$$\sigma_T(\bar{f},\chi) := \begin{cases} q^{-n}\mathrm{Card}\{\bar{t} \in \overline{T} \mid \bar{t} \text{ is a non singular root of } \bar{f}\} & \textit{if } \chi = \chi_{triv} \\ 0 & \textit{if } \chi \neq \chi_{triv}. \end{cases}$$

Denote by $Z_T(s,f,\chi)$ the integral $\int_T \chi(ac\ f(x))\ |f(x)|^s\ |dx|$.

Lemma 1.2.3 ([37, Igusa's Stationary Phase Formula])**.** *With all the notation above we have*

$$Z_T(s,f,\chi) = \nu_T(\overline{f},\chi) + \sigma_T(\overline{f},\chi)\frac{(1-q^{-1})q^{-s}}{(1-q^{-1-s})} + \int_{S_T(f)} \chi(ac\ f(x))\ |f(x)|^s\ |dx|,$$

where $Re(s) > 0$.

Lemma 1.2.4 ([37, [Lemma 2.4])**.** *Let $T \subseteq O_K^n$ be the preimage under the canonical homomorphism $O_K \to O_K/P_K$ of a subset $\overline{T} \subseteq \mathbb{F}_q^n$. Let $f(x) \in O_K[x]$ be a polynomial such that $Sing_f(K) \cap T = \emptyset$, then*

$$\int_T \chi(ac\ f(x))\ |f(x)|^s\ |dx| = \begin{cases} \frac{L_1(q^{-s})}{1-q^{-1}\ q^{-s}} & \textit{if } \chi = \chi_{triv}, \\ \\ L_2(q^{-s}) & \textit{if } \chi \neq \chi_{triv}, \end{cases}$$

where $L_1(q^{-s}), L_2(q^{-s}) \in \mathbb{Q}[q^{-s}]$.

Now we might mention the following result, which is essential to obtain asymptotic expansions for exponential sums attached to certain polynomials, as we will see in Chapter 2.

We recall here that the *critical set* of f is defined as

$$C_f := C_f(K) = \{(x, y) \in K^2 \mid \nabla f(x, y) = 0\}.$$

Theorem 1.2.1 ([22, [Lemma 8.4.1]). *Assume that* $char(K) = 0$ *and* C_f *is contained in* $f^{-1}(0)$. *Then there exists* $e > 0$ *in* $\mathbb{N}$, *such that* $Z(s, f, \chi) = 0$ *unless* $c_\chi \leqslant e$, *for* $\chi = \omega|_{O_K^\times}$.

1.2.2 Exponential Sums mod $\mathfrak{p}^m$

We recall that for a given $z = \sum_{n=n_0}^{\infty} z_n p^n \in \mathbb{Q}_p$, with $z_n \in \{0, \ldots, p-1\}$ and $z_{n_0} \neq 0$, the *fractional part* of z is

$$\{z\}_p := \begin{cases} 0 & \text{if } n_0 \geq 0 \\ \\ \sum_{n=n_0}^{-1} z_n p^n & \text{if } n_0 < 0. \end{cases}$$

Then for $z \in \mathbb{Q}_p$, $\exp(2\pi\sqrt{-1}\,\{z\}_p)$, is an *additive character* on $\mathbb{Q}_p$, which is trivial on $\mathbb{Z}_p$ but not on $p^{-1}\mathbb{Z}_p$.

If $Tr_{K/\mathbb{Q}_p}(\cdot)$ denotes the trace function of the extension, then there exists an integer $d \geq 0$ such that $Tr_{K/\mathbb{Q}_p}(z) \in \mathbb{Z}_p$ for $|z| \leq q^d$ but $Tr_{K/\mathbb{Q}_p}(z_0) \notin \mathbb{Z}_p$ for some z_0 with $|z_0| = q^{d+1}$. d is known as *the exponent of the different* of $K/\mathbb{Q}_p$ and by, e.g. [33, Chap.

VIII, Corollary of Proposition 1] $d \geq e-1$, where e is the ramification index of $K/\mathbb{Q}_p$. For $z \in K$, the additive character

$$\varkappa(z) = \exp(2\pi\sqrt{-1}\left\{Tr_{K/\mathbb{Q}_p}(\mathfrak{p}^{-d}z)\right\}_p),$$

is *a standard character* of K, i.e. $\varkappa$ is trivial on O_K but not on $\mathfrak{p}^{-1}O_K$. In our case, it is more convenient to use

$$\Psi(z) = \exp(2\pi\sqrt{-1}\left\{Tr_{K/\mathbb{Q}_p}(z)\right\}_p),$$

instead of $\varkappa(\cdot)$, since we will use Denef's approach for estimating exponential sums, see Proposition (3.1.1) below.

Now, let K be a local field of characteristic $p > 0$, i.e. $K = \mathbb{F}_q((T))$. Take

$$z(T) = \sum_{i=n_0}^{\infty} z_i T^i \in K,$$

we define $Res(z(T)) := z_{-1}$. Then one may see that

$$\Psi(z(T)) := \exp(2\pi\sqrt{-1}\ Tr_{\mathbb{F}_q/\mathbb{F}_p}(Res(z(T)))),$$

is a standard additive character on K.

Fixing an additive character $\Psi : K \to \mathbb{C}$, the exponential sums $\mod \mathfrak{p}^m$ attached to f is defined as

$$E(z,f) = \int_{O_K^n} \Psi(zf(x))\ |dx|,$$

where $z = \mathfrak{p}^{-m}u,\ u \in O_K^{\times}$.

Notice that

$$\int_{O_K^n} \Psi(zf(x))\ |dx| = \sum_{\tilde{x}\in(O_K/P_K^m)^n} q^{-mn}\ \Psi(zf(\tilde{x}))\ |dx|.$$

A central mathematical problem consists in describing the asymptotic behavior of $E(z, f)$ as $|z| \to \infty$, this task can be a accomplished by using for instance the following Proposition, see Corollary 1.4.5 of [12]. See also our results about this mater in the case of arithmetically non-degenerate polynomials (Theorem 3.1.1).

We denote by $\mathrm{Coeff}_{t^k} Z(s, f, \chi)$ the coefficient c_k in the power series expansion of $Z(s, f, \chi)$ in the variable $t = q^{-s}$.

Proposition 1.2.1 ([12, Proposition 1.4.4])**.** *Let $u \in O_K^\times$ and $m \in \mathbb{Z}$. Then $E(u\mathfrak{p}^{-m})$ equals*

$$Z(0, \chi_{triv}) + Coeff_{t^{m-1}} \frac{(t-q)Z(s, \chi_{triv})}{(q-1)(1-t)} + \sum_{\chi \neq \chi_{triv}} g_{\chi^{-1}} \chi(u) Coeff_{t^{m-c(\chi)}} Z(s, \chi),$$

where $c(\chi)$ denotes the conductor of χ, i.e. the smallest $c \geqslant 1$ such that χ is trivial on $1 + P_K^c$ and g_χ is the Gaussian sum

$$g_\chi = (q-1)^{-1} q^{1-c(\chi)} \sum_{x \in (O_K/P_K^{c(\chi)})^\times} \chi(v)\ \Psi(v/\mathfrak{p}^{c(\chi)}).$$

1.3 Newton's polyhedron and non-degeneracy conditions

There exists a generic class of polynomials named *non-degenerated with respect to its Newton Polyhedron* for which is possible to give a small set of candidates for the poles of $Z(s, f)$. For sake of completeness, we review some basic notions and well-known results about Newton polyhedron and non-degenerated polynomials, see e.g [14], for this reason we do not give proofs.

Definition 1.3.1. *Given a non-constant polynomial $f(x) = \sum_l a_l x^l \in K[x]$, for $x = (x_1, \cdots x_n)$, satisfying $f(0) = 0$, we define the* support *of f as: $Supp(f) = \{l \in \mathbb{N}^n; a_l \neq$*

$0\}$, *and the geometric Newton polyhedron* $\Gamma^{geom}(f)$ *of* f *as:*

$$\Gamma^{geom}(f) := ConvexHull\{\bigcup_{l\in Supp(f)} (l + \mathbb{R}^n_{\geqslant 0})\}.$$

A face of $\Gamma^{geom}(f)$ of codimension 1 is named a *facet*. Each facet is lying on an affine hyperplane of the form $\sum_i a_{i,j}x^i = m(a_j)$, where a_j is a vector whose coordinates are positive integers. Note that each proper face τ of $\Gamma^{geom}(f)$ is the finite intersection of the facets of $\Gamma^{geom}(f)$ which contain τ.

Definition 1.3.2. *Let* f *be as in definition 1.3.1. For every face* τ *of* $\Gamma^{geom}(f)$*, we define the function face*

$$f_\tau = \sum_{l\in\tau} a_l x^l.$$

We set $\langle\cdot\rangle$ for the usual inner product in $\mathbb{R}^n$ and identify the dual vector space with $\mathbb{R}^n$.

Definition 1.3.3. *For* $a \in \mathbb{R}^n$*, we define* $m(a) = \inf_{x\in\Gamma^{geom}(f)}\{\langle a\cdot x\rangle\}$ *and the first meet locus of* a *as*

$$F(a) = \{x \in \Gamma^{geom}(f)|\langle a\cdot x\rangle = m(a)\},$$

where $a\cdot x$ *denotes the scalar product* $\sum_i^n a_i x^i$ *of* $a = (a_1, \cdots, a_n)$ *and* $x = (x_1, \cdots, x_n)$.

Now we define an equivalence relation on $\mathbb{R}^n$ by $a \sim a'$ if and only if $F(a) = F(a')$. In particular $F(0) = \Gamma^{geom}(f)$ and $F(a)$ is a proper face of $\Gamma^{geom}(f)$, if $a \neq 0$. Moreover $F(a)$ is a compact face if and only if $a \in \mathbb{R}^n_+$. A vector $a \in \mathbb{R}^n$ is called primitive if the components of a are integers whose greatest common divisor is one. Furthermore for every facet of $\Gamma^{geom}(f)$ there exist a unique primitive vector in $\mathbb{N}^n \setminus \{0\}$, which is perpendicular to that facet.

We will give a selection of some definitions and properties of a polyhedral subdivision of $\mathbb{R}^n$.

If τ is a face of $\Gamma^{geom}(f)$, we define the cone associated to τ as

$$\Delta_\tau = \{a \in \mathbb{R}^n_+|\ F(a) = \tau\}.$$

Let $\gamma_1, \cdots, \gamma_n$ be the facets of $\Gamma^{geom}(f)$ containing τ, and let $a_1, \cdots, a_n$ be the orthogonal vectors to $\gamma_1, \cdots, \gamma_n$ respectively. Then one proves that $\mathbb{R}_{\geqslant 0} \setminus \{(0, \cdots, 0)\}$ is the disjoint union of the $\Delta_\tau = \{\lambda a_1 + \cdots + \lambda a_n \mid \lambda_1, \cdots, \lambda_n \in \mathbb{R}_{>0}\}$, and its dimension is equal to $n - \dim \tau$. This gives the geometry of the other equivalence classes Δ_τ. It is well-known that the closure of Δ is $\overline{\Delta} := \{a \in \mathbb{R}_+^n : F(a) \supset \tau\} = \{\lambda_1 a_1 + \cdots + \lambda_e a_e : \lambda_i \in \mathbb{R},\ \lambda_i \geqslant 0\}$.

Definition 1.3.4. *If $a_1, \cdots, a_e \in \mathbb{R}^n \setminus \{0\}$, we call $\{\lambda_1 a_1 + \cdots + \lambda_e a_e : \lambda_i \in \mathbb{R},\ \lambda_i > 0\}$, the cone strictly positively spanned by the vectors $a_1, \cdots, a_e$. Suppose a cone Δ is strictly positively spanned by vectors $a_1, \cdots, a_e \in \mathbb{R}^n \setminus \{0\}$. If $a_1, \cdots, a_e$ are linearly independent over $\mathbb{R}$, Δ is called a simplicial cone. If moreover $a_1, \cdots, a_e \in \mathbb{Z}^n$, we say Δ is a rational simplicial cone. If $\{a_1, \cdots, a_e\}$ is a subset of a basis of the $\mathbb{Z}$- module $\mathbb{Z}^n$, we call Δ a simple cone.*

Remark 1.3.1.

1. *One can divide the cone Δ_τ associated to τ into a finite number of rational simplicial cones such that each Δ_i in the subdivision is spanned by vectors from the set $\{a_1, \cdots, a_e\}$, without introducing new rays.*

2. *One can even find a partition of Δ_τ into simple cones, but in general it will then be necessary to introduce new generators.*

Summarizing given a polynomial $f(x) \in K[x], f(0) = 0$, with Newton polyhedron $\Gamma^{geom}(f)$, there exists a finite partition of $\mathbb{R}_+^n$ of the form:

$$\mathbb{R}_+^n = \{(0, \cdots 0)\} \cup \bigcup_i \Delta_i,$$

where each Δ_i is a simplicial cone contained in an equivalence class of $\simeq$. Moreover, by Remark 1.3.1, it is possible to refine this partition in such a way that each Δ_i is a simple cone contained in an equivalence class of $\simeq$.

Once we have a simplicial conical subdivision subordinated to $\Gamma^{geom}(f)$, it is possible to reduce the computation of $Z(s, f, \chi)$ to integrals over the cones in Δ_τ. In order to do that let $f(x) \in K[x]$ be a non-constant polynomial satisfying $f(0) = 0$, and let $\Gamma^{geom}(f)$

be its Newton polyhedron . We fix a simplicial conical subdivision $\{\Delta_\tau\}_{\tau\subset\Gamma^{geom}(f)}$ of $\mathbb{R}_+^n$ subordinated to $\Gamma^{geom}(f)$, we set

$$\begin{aligned} E_{\Delta_\tau} &:= \{(x_1,\cdots,x_n)\in O_K^n \mid (v(x_1),\cdots,v(x_n))\in\Delta_\tau\},\\ Z(s,f,\chi,\Delta_\tau) &:= \int_{E_{\Delta_\tau}} \chi(ac\ f(x))|f(x)|^s\ |dx|,\ \textit{and}\\ Z(s,f,\chi,O_K^{\times n}) &:= \int_{O_K^{\times n}} \chi(ac\ f(x))|f(x)|^s\ |dx|. \end{aligned}$$

Therefore we have that,

$$Z(s,f,\chi) = Z(s,f,\chi,O_K^{\times n}) + \sum_{\tau\subset\Gamma^{geom}(f)} Z(s,f,\chi,\Delta_\tau). \tag{1.3.1}$$

The following definition plays a relevant role in the study of local zeta functions by Newton polyhedron techniques.

Definition 1.3.5. *A non-constant polynomial f, satisfying $f(0)=0$, is called* non-degenerated with respect to its Newton polyhedron $\Gamma^{geom}(f)$ *in the sense of Kouchnirenko, if for each compact face τ of $\Gamma^{geom}(f)$*

$$f_\tau(x_1,\cdots,x_n) = \frac{\partial f_\tau}{\partial x_1} = \frac{\partial f_\tau}{\partial x_2} = \cdots = \frac{\partial f_\tau}{\partial x_n} = 0,$$

has no solution in $(K\setminus\{0\})^n$.

There are other variations of the same condition, see e.g [14].

We now show that the stationary phase formula gives a small set of candidates for the poles of $Z(s,f,\chi)$ in terms of the Newton polyhedron $\Gamma^{geom}(f)$.

Theorem 1.3.1 ([37, [Theorem A]). *Let K be a non-Archimedean local field, and let $f(x)\in O_K[x]$ be a polynomial globally non-degenerate with respect to its Newton polyhedron $\Gamma^{geom}(f)$. Then the Igusa local zeta functions $Z(s,f,\chi)$ is a rational function of q^{-s} satisfying:*

1. *if s is a pole of $Z(s,f,\chi)$, then*

$$s = -\frac{|a_\gamma|}{m(a_\gamma)} + \frac{2\pi}{\log q}\ \frac{k}{m(a_\gamma)}, k \in \mathbb{Z}$$

   for some facet γ of $\Gamma^{geom}(f)$ with perpendicular a_γ, and $m(a_\gamma) \neq 0$, or

$$s = -1 + \frac{2\pi}{\log q}k,\ \ k \in \mathbb{Z};$$

2. *if $\chi \neq \chi_{triv}$ and the order of χ does not divide any $m(a_\gamma) \neq 0$, where γ is a facet of $\Gamma^{geom}(f)$, then $Z(s,f,\chi)$ is a polynomial in q^{-s}, and its degree is bounded by a constant independent of χ.*

1.3.1 Example

Example 1.3.1. *Let $f(x,y) = (y^3 - x^2)^2 + x^4y^4 \in K[x,y]$. We assume that the characteristic of the residue field of K is different from 2 and 3. Note that, the support of $f(x,y)$ is Supp$(f) = \{(4,0),(2,3),(4,4),(0,6)\}$. Note also that the origin of K^2 is the only singular point of f and this polynomial is degenerate with respect to $\Gamma^{geom}(f)$.*

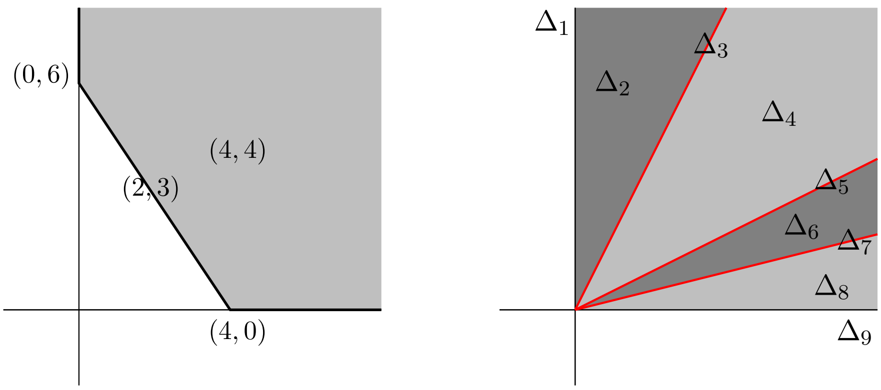

Figure 1.1: $\Gamma^{geom}((y^3 - x^2)^2 + x^4y^4)$ and the conical partition of $\mathbb{R}^2_+$ induced by it.

Now, the simple conical subdivision of $\mathbb{R}^2_+$ subordinated to the geometric Newton *polygon of* $f(x,y)$ *is* $\mathbb{R}_+^2 = \{(0,0)\} \cup \bigcup_{j=1}^{9} \Delta_j$, *where the* Δ_j *are in Table 1.1.*

Cone	Generators
Δ_1	$(0,1)\mathbb{R}_+ \setminus \{0\}$
Δ_2	$(0,1)\mathbb{R}_+ \setminus \{0\} + (1,1)\mathbb{R}_+ \setminus \{0\}$
Δ_3	$(1,1)\mathbb{R}_+ \setminus \{0\}$
Δ_4	$(1,1)\mathbb{R}_+ \setminus \{0\} + (3,2)\mathbb{R}_+ \setminus \{0\}$
Δ_5	$(3,2)\mathbb{R}_+ \setminus \{0\}$
Δ_6	$(3,2)\mathbb{R}_+ \setminus \{0\} + (2,1)\mathbb{R}_+ \setminus \{0\}$
Δ_7	$(2,1)\mathbb{R}_+ \setminus \{0\}$
Δ_8	$(2,1)\mathbb{R}_+ + (1,0)\mathbb{R}_+$
Δ_9	$(1,0)\mathbb{R}_+ \setminus \{0\}$

Table 1.1: Conical subdivision of $\mathbb{R}_+^2 \setminus \{(0,0)\}$

1.3.2 An explicit formula for $Z(s,f,\chi)$

There is another proof of the fact that $Z(s,f,\chi)$ is a rational function of q^{-s}. In [14] the authors provide a formula for $Z(s,f,\chi)$ that holds if f is non-degenerated over $\mathbb{F}_q$ with respect to all the faces of its Newton polyhedron and if the conductor c_χ of χ is equal to 1.

Theorem 1.3.2. [14] *Let p be a prime number. Let f be as in definition 1.3.1. Suppose that f is non-degenerated over the finite field $\mathbb{F}_q$ with respect to all the faces of its Newton polyhedron $\Gamma^{geom}(f)$. Let χ be a character of $\mathbb{Z}_p^\times$ with conductor $c_\chi = 1$. Denote for each face τ of $\Gamma^{geom}(f)$ by N_τ the number of elements in the set*

$$\{a \in (\mathbb{F}_q)^n \mid \overline{f}_\tau(a) = 0\}.$$

Let s be a complex variable with $Re(s) > 0$. Then $Z(s,f,\chi) = \sum_{\tau \in \Gamma^{geom}(f)} L_\tau S_{\Delta_\tau}$, *with*

$$L_\tau = \begin{cases} q^{-n}((q-1)^n - qN_\tau \frac{q^s-1}{q^{s+1}-1}) & \text{for } \chi = \chi_{triv}, \\ q^{-n} \sum_{a \in (\mathbb{F}_q^\times)^n} \chi(f_\tau(a)) & \text{for } \chi \neq \chi_{triv}, \end{cases}$$

and $S_{\Delta_\tau} = \sum_{k\in\mathbb{N}^n\cap\Delta_\tau} q^{-\sigma(k)-m(k)s,}$ *for each face* τ *of* $\Gamma^{geom}(f)$ *(including* $\tau = \Gamma^{geom}(f)$*), with* $\sigma(k) = k_1, \cdots, k_n$*, and* $m(k)$ *as in defintion 1.3.3.*

We have $S_{\Delta_{\Gamma^{geom}(f)}} = 1$ *and the other* S_{Δ_τ}*, can be calculated as follows. Take a partition of the cone* Δ_τ *associated to the proper face* τ *into rational simplicial cones* Δ_i*. Then clearly* $S_{\Delta_\tau} = \sum_i S_\Delta$*, where the summation is over the rational simplicial cones* Δ_i *and*

$$S_{\Delta_i} = \sum_{k\in\mathbb{N}^n\cap\Delta_i} q^{\sigma(k)-m(k)s}.$$

Let Δ_i *be the cone strictly positively spanned by the linearly independent vectors* $a_1, \cdots, a_r \in \mathbb{N}^\setminus\{0\}$*. Then*

$$S_{\Delta_i} = \frac{\sum_h q^{\sigma(h)+m(h)s}}{(q^{\sigma(a_1)+m(a_1)s} - 1)\cdots(q^{\sigma(a_r)+m(a_r)s} - 1)},$$

where h *runs through the elements of the set*

$$\mathbb{Z}^n \cap \left\{ \sum_{j=1}^{r} \lambda_j a_j \mid 0 \leqslant \lambda_j < 1, \textit{for } j = 1, \cdots, r \right\}.$$

Remark 1.3.2. *1. Clearly* S_{Δ_τ} *is a rational function in* q^{-s} *for* $s \in \mathbb{C}$*.*

2. Note that L_τ *depend on the coefficients of the polynomial* f *and is a rational function in* q^{-s} *for* $s \in \mathbb{C}$*.*

Chapter 2

Igusa's Local Zeta Functions for Arithmetically Non Degenerate Polynomials

In this chapter we study the twisted local zeta function associated to a polynomial in two variables with coefficients in a non–Archimedean local field of arbitrary characteristic. Under the hypothesis that the polynomial is arithmetically non degenerate, we obtain an explicit list of candidates for the poles in terms of geometric data obtained from a family of arithmetic Newton polygons attached to the polynomial, see Theorem 2.4.1. The notion of arithmetical non degeneracy due to Saia and Zúñiga-Galindo is weaker than the usual notion of non degeneracy due to Kouchnirenko, see Section 2.2. This chapter is an extended version of the results in [1].

2.1 Arithmetic Newton Polygons and Non-Degeneracy Conditions.

2.1.1 Semi–quasihomogeneous polynomials

Let L be a field, and a, b two coprime positive integers. A polynomial $f(x,y) \in L[x,y]$ is called quasihomogeneous with respect to the weight (a,b) if it has the form

$$f(x,y) = cx^u y^v \prod_{i=1}^{l} (y^a - \alpha_i x^b)^{e_i}, c \in L^{\times}.$$

Note that such a polynomial satisfies $f(t^a x, t^b y) = t^d f(x,y)$, for every $t \in L^{\times}$, and thus this definition of quasihomogeneity coincides with the standard one after a finite extension of L. The integer d is called the weighted degree of $f(x,y)$ with respect to (a,b).

A polynomial $f(x,y)$ is called *semi–quasihomogeneous* with respect to the weight (a,b) when

$$f(x,y) = \sum_{j=0}^{l_f} f_j(x,y), \tag{2.1.1}$$

and the $f_j(x,y)$ are quasihomogeneous polynomials of degree d_j with respect to (a,b), and $d_0 < d_1 < \cdots < d_{l_f}$. The polynomial $f_0(x,y)$ is called the *quasihomogeneous tangent cone* of $f(x,y)$.

We set

$$f_j(x,y) := c_j x^{u_j} y^{v_j} \prod_{i=1}^{l_j} (y^a - \alpha_{i,j} x^b)^{e_{i,j}}, \quad c_j \in L^{\times}.$$

We assume that d_j is the weighted degree of $f_j(x,y)$ with respect to (a,b), thus

$$d_j := ab \left(\sum_{i=1}^{l_j} e_{i,j} \right) + au_j + bv_j.$$

Now, let $f(x,y) \in L[x,y]$ be a semi-quasihomogeneous polynomial of the form (2.1.1), and take $\theta \in L^\times$ a fixed root of $f_0(1,y^a)$. We put $e_{j,\theta}$ for the multiplicity of θ as a root of $f_j(1,y^a)$. To each $f_j(x,y)$ we associate a straight line of the form

$$w_{j,\theta}(z) := (d_j - d_0) + e_{j,\theta} z, \quad j = 0, 1, \cdots, l_f,$$

where z is a real variable.

Definition 2.1.1. *1. The arithmetic Newton polygon $\Gamma_{f,\theta}$ of $f(x,y)$ at θ is*

$$\Gamma_{f,\theta} = \{(z,w) \in \mathbb{R}_+^2 \mid w \leqslant \min_{0 \leqslant j \leqslant l_f} \{w_{j,\theta}(z)\}\}.$$

2. The arithmetic Newton polygon $\Gamma^A(f)$ of $f(x,y)$ is defined as the family

$$\Gamma^A(f) = \{\Gamma_{f,\theta} \mid \theta \in L^\times, \ f_0(1,\theta^a) = 0\}.$$

If $\mathcal{Q} = (0,0)$ or if $\mathcal{Q}$ is a point of the topological boundary of $\Gamma_{f,\theta}$ which is the intersection point of at least two different straight lines $w_{j,\theta}(z)$, then we say that $\mathcal{Q}$ is a *vertex* of $\Gamma^A(f)$. The boundary of $\Gamma_{f,\theta}$ is formed by r straight segments, a half–line, and the non–negative part of the horizontal axis of the $(w,z)-$plane. Let $\mathcal{Q}_k, k = 0, 1, \cdots, r$ denote the vertices of the topological boundary of $\Gamma_{f,\theta}$, with $\mathcal{Q}_0 := (0,0)$. Then the equation of the straight segment between $\mathcal{Q}_{k-1}$ and $\mathcal{Q}_k$ is

$$w_{k,\theta}(z) = (\mathcal{D}_k - d_0) + \varepsilon_k z, \quad k = 1, 2, \cdots, r. \tag{2.1.2}$$

The equation of the half–line starting at $\mathcal{Q}_r$ is,

$$w_{r+1,\theta}(z) = (\mathcal{D}_{r+1} - d_0) + \varepsilon_{r+1} z. \tag{2.1.3}$$

Therefore

$$\mathcal{Q}_k = (\tau_k, (\mathcal{D}_k - d_0) + \varepsilon_k \tau_k), \quad k = 1, 2, \cdots r, \tag{2.1.4}$$

where $\tau_k := \frac{(\mathcal{D}_{k+1}-\mathcal{D}_k)}{\varepsilon_k-\varepsilon_{k+1}} > 0, \quad k = 1, 2, \cdots r$. Note that $\mathcal{D}_k = d_{j_k}$ and $\varepsilon_k = e_{j_k,\theta}$, for some index $j_k \in \{1, \ldots, l_j\}$. In particular, $\mathcal{D}_1 = d_0$, $\varepsilon_1 = e_{0,\theta}$, and the first equation is $w_{1,\theta}(z) = \varepsilon_1 z$. If $\mathcal{Q}$ is a vertex of the boundary of $\Gamma_{f,\theta}$, the *face function* is the polynomial

$$f_{\mathcal{Q}}(x, y) := \sum_{w_{j,\theta}(\mathcal{Q})=0} f_j(x, y), \tag{2.1.5}$$

where $w_{j,\theta}(z)$ is the straight line corresponding to $f_j(x, y)$.

Definition 2.1.2. *1. A semi-quasihomogeneous polynomial $f(x, y) \in L[x, y]$ is called arithmetically non-degenerate modulo $\mathfrak{p}$ with respect to $\Gamma_{f,\theta}$ at θ, if the following conditions holds.*

(a) The origin of $\mathbb{F}_q^2$ is a singular point of $\overline{f}$, i.e. $\overline{f}(0, 0) = \nabla\overline{f}(0, 0) = 0$;

(b) $\overline{f}(x, y)$ does not have singular points on $(\mathbb{F}_q^\times)^2$;

(c) for any vertex $\mathcal{Q} \neq \mathcal{Q}_0$ of the boundary of $\Gamma_{f,\theta}$, the system of equations

$$\overline{f}_{\mathcal{Q}}(x, y) = \frac{\partial\overline{f}_{\mathcal{Q}}}{\partial x}(x, y) = \frac{\partial\overline{f}_{\mathcal{Q}}}{\partial y}(x, y) = 0,$$

has no solutions on $(\mathbb{F}_q^\times)^2$.

2. If a semi–quasihomogeneous polynomial $f(x, y) \in L[x, y]$ is arithmetically non-degenerate with respect to $\Gamma_{f,\theta}$, for each $\theta \in L^\times$ satisfying $f_0(1, y^a) = 0$, then $f(x, y)$ is called arithmetically non-degenerate with respect to $\Gamma^A(f)$.

2.2 Arithmetically non-degenerate polynomials

Let $a_\gamma = (a_1(\gamma), a_2(\gamma))$ be the normal vector of a fixed edge γ of $\Gamma^{geom}(f)$. It is well known that $f(x, y)$ is a semi–quasihomogeneous polynomial with respect to the weight a_γ, in this case we write

$$f(x, y) = \sum_{j=0}^{l_f} f_j^\gamma(x, y),$$

where $f_j^\gamma(x,y)$ are quasihomogeneous polynomials of degree $d_{j,\gamma}$ with respect to a_γ, cf. (2.1.1). We define

$$\Gamma_\gamma^A(f) = \{\Gamma_{f,\theta} \mid \theta \in L^\times,\ f_0^\gamma(1,\theta^{a_1(\gamma)}) = 0\},$$

i.e. this is the arithmetic Newton polygon of $f(x,y)$ regarded as a semi quasihomogeneous polynomial with respect to the weight a_γ. Then we define

$$\Gamma^A(f) = \bigcup_{\gamma \text{ edge of } \Gamma^{geom}(f)} \Gamma_\gamma^A(f).$$

Definition 2.2.1. $f(x,y) \in L[x,y]$ *is called arithmetically non-degenerate modulo* $\mathfrak{p}$ *with respect to its arithmetic Newton polygon, if for every edge* γ *of* $\Gamma^{geom}(f)$, *the semi–quasihomogeneous polynomial* $f(x,y)$, *with respect to the weight* a_γ, *is arithmetically non-degenerate modulo* $\mathfrak{p}$ *with respect to* $\Gamma_\gamma^A(f)$.

2.3 Examples

In this section we show two examples to illustrate the geometric ideas presented in the previous sections.

2.3.1 The local zeta function of $(y^3 - x^2)^2 + x^4y^4$

This examples are adapted to our case from [27]. We obtain an explicit list of candidates for the poles in terms of geometric data obtained from a family of arithmetic Newton polygons attached to the polynomial in each example.

Computation of $Z(s,f,\chi,\Delta_i), i = 1,2,3,4,6,7,8,9.$

These integrals correspond to the case in which f is non–degenerate on Δ_i. We show the Newton polygon and the correspond conical subdivision of $\mathbb{R}_+^2$ in the figure 1.1 of the example 1.3.1.

The integral corresponding to Δ_3, can be calculated as follows.

$$Z(s,f,\chi,\Delta_3) = \sum_{n=1}^{\infty} \int_{\mathfrak{p}^n O_K^{\times} \times \mathfrak{p}^n O_K^{\times}} \chi(ac\ f(x,y))|f(x,y)|^s|dxdy|$$

$$= \sum_{n=1}^{\infty} q^{-2n-4ns} \int_{O_K^{\times 2}} \chi(ac\ (\mathfrak{p}^n y^3 - x^2)^2 + \mathfrak{p}^{4n}x^4y^4)|(\mathfrak{p}^n y^3 - x^2)^2 + \mathfrak{p}^{4n}x^4y^4|^s|dxdy|.$$

We set $g_3(x,y) = (\mathfrak{p}^n y^3 - x^2)^2 + \mathfrak{p}^{4n}x^4y^4$, then $\overline{g}_3(x,y) = x^4$ and the origin is the only singular point of $\overline{g}_3$. We decompose $O_K^{\times 2}$ as

$$O_K^{\times 2} = \bigsqcup_{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2} (a,b) + (\mathfrak{p}O_K)^2,$$

thus

$$Z(s,f,\chi,\Delta_3) = \sum_{n=1}^{\infty} q^{-2n-4ns} \sum_{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2} \int_{(a,b)+(\mathfrak{p}O_K)^2} \chi(ac\ g_3(x,y))|g_3(x,y)|^s|dxdy|$$

$$= \sum_{n=1}^{\infty} q^{-2n-4ns-2} \sum_{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2} \int_{O_K^2} \chi(ac\ g_3(a+\mathfrak{p}x, b+\mathfrak{p}y))|g_3(a+\mathfrak{p}x, b+\mathfrak{p}y)|^s|dxdy|.$$

Now, by using the Taylor series for g around (a,b):

$$g(a+\mathfrak{p}x, b+\mathfrak{p}y) = g(a,b) + \mathfrak{p}\left(\frac{\partial g}{\partial x}(a,b)x + \frac{\partial g}{\partial y}(a,b)y\right) + \mathfrak{p}^2(\textit{higher order terms}),$$

and the fact that $\frac{\partial \overline{g_3}}{\partial x}(\overline{a},\overline{b}) = 4\overline{a}^3 \not\equiv 0 \mod \mathfrak{p}$, we can change variables in the previous integral as follows

$$\begin{cases} z_1 = \frac{g_3(a+\mathfrak{p}x, b+\mathfrak{p}y) - g_3(a,b)}{\mathfrak{p}} \\ z_2 = y. \end{cases} \tag{2.3.1}$$

This transformation gives a bianalytic mapping on O_K^2 that preserves the Haar measure.

Hence by Lemma 1.2.2, we get

$$Z(s,f,\chi,\Delta_3)=$$

$$\sum_{n=1}^{\infty} q^{-2n-4ns-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int_{O_K} \chi(ac\ (g_3(a,b)+\mathfrak{p}z_1))|g_3(a,b)+\mathfrak{p}z_1)|^s\ |dz_1|,$$

$$=\begin{cases} \frac{q^{-2-4s}(1-q^{-1})^2}{(1-q^{-2-4s})} & \textit{if}\ \ \chi=\chi_{triv} \\ \frac{q^{-2-4s}(1-q^{-1})^2}{(1-q^{-2-4s})} & \textit{if}\ \ \chi^4=\chi_{triv}, \chi|_U=\chi_{triv} \\ 0 & \textit{all other cases}, \end{cases}$$

where $U=1+\mathfrak{p}O_K$.

We note here that for $i=1,2,4,6,7,8$ and 9, the computation of the $Z(s,f,\chi,\Delta_i)$ are similar to the case $Z(s,f,\chi,\Delta_3)$.

Computation of $\mathbf{Z(s,f,\chi,\Delta_5)}$ (An integral on a degenerate face in the sense of Kouchnirenko)

$$Z(s,f,\chi,\Delta_5)=\sum_{n=1}^{\infty}\int_{\mathfrak{p}^{3n}O_K^\times\times\mathfrak{p}^{2n}O_K^\times} \chi(ac\ f(x,y))\ |f(x,y)|^s|dxdy|, \tag{2.3.2}$$

$$=\sum_{n=1}^{\infty} q^{-5n-12ns}\int_{O_K^{\times 2}} \chi(ac((y^3-x^2)^2+\mathfrak{p}^{8n}x^4y^4))|(y^3-x^2)^2+\mathfrak{p}^{8n}x^4y^4|^s\ |dxdy|.$$

Let $f^{(n)}(x,y)=(y^3-x^2)^2+\mathfrak{p}^{8n}x^4y^4$, for $n\geqslant 1$. We define

$$\begin{aligned} \Phi:\ \ O_K^{\times 2} &\longrightarrow O_K^{\times 2} \\ (x,y) &\longmapsto (x^3y, x^2y). \end{aligned} \tag{2.3.3}$$

Φ is an analytic bijection of $O_K^{\times 2}$ onto itself that preserves the Haar measure, so it can be used as a change of variables in (2.3.2). We have $(f^{(n)}\circ\Phi)(x,y)=x^{12}y^4\widetilde{f^{(n)}}(x,y)$, with $\widetilde{f^{(n)}}(x,y) = (y-1)^2 + \mathfrak{p}^{8n}x^8y^4$, and then

$$I(s, f^{(n)}, \chi) :=$$
$$\int_{O_K^{\times 2}} \chi(ac((y^3-x^2)^2 + \mathfrak{p}^{8n}x^4y^4))\ |(y^3-x^2)^2 + \mathfrak{p}^{8n}x^4y^4|^s\ |dxdy|,$$
$$= \int_{O_K^{\times 2}} \chi(ac(x^{12}y^4\widetilde{f^{(n)}}(x,y)))\ |\widetilde{f^{(n)}}(x,y)|^s\ |dxdy|.$$

Now, we decompose $O_K^{\times 2}$ as follows:

$$O_K^{\times 2} = \left(\bigsqcup_{y_0 \not\equiv 1 \bmod \mathfrak{p}} O_K^{\times} \times \{y_0 + \mathfrak{p}O_K\} \right) \bigcup O_K^{\times} \times \{1 + \mathfrak{p}O_K\},$$

where y_0 runs through a set of representatives of $\mathbb{F}_q^{\times}$ in O_K. By using this decomposition,

$$I(s, f^{(n)}, \chi) =$$
$$\sum_{y_0 \not\equiv 1 \bmod \mathfrak{p}} \sum_{j=0}^{\infty} q^{-1-j} \int_{O_K^{\times 2}} \chi(ac(x^{12}[y_0 + \mathfrak{p}^{j+1}y]^4 \widetilde{f^{(n)}}(x, y_0 + \mathfrak{p}^{j+1}y)))\ |dxdy|$$
$$+ \sum_{j=0}^{\infty} q^{-1-j} \int_{O_K^{\times 2}} \mathcal{X}(x^{12}[1 + \mathfrak{p}^{j+1}y]^4 \widetilde{f^{(n)}}(x, 1 + \mathfrak{p}^{j+1}y))\ |dxdy|,$$

where

$$\mathcal{X}(x^{12}[1 + \mathfrak{p}^{j+1}y]^4 \widetilde{f^{(n)}}(x, 1 + \mathfrak{p}^{j+1}y)) =$$
$$\chi(x^{12}[1 + \mathfrak{p}^{j+1}y]^4 \widetilde{f^{(n)}}(x, 1 + \mathfrak{p}^{j+1}y)) \times |x^{12}[1 + \mathfrak{p}^{j+1}y]^4 \widetilde{f^{(n)}}(x, 1 + \mathfrak{p}^{j+1}y)|^s.$$

Finally,

$$I(s,f^{(n)},\chi)=\sum_{y_0\not\equiv 1 \bmod \mathfrak{p}}\sum_{j=0}^{\infty}q^{-1-j}\int\limits_{O_K^{\times 2}}\chi(ac(f_1(x,y)))\ |dxdy|$$
$$+\sum_{j=0}^{4n-2}q^{-1-j-(2+2j)s}\int\limits_{O_K^{\times 2}}\chi(ac(f_2(x,y)))\ |dxdy|$$
$$+q^{-4n-8ns}\int\limits_{O_K^{\times 2}}\chi(f_3(x,y))\ |f_3(x,y)|^s\ |dxdy|$$
$$+\sum_{j=4n}^{\infty}q^{-j-1-8ns}\int\limits_{(O_K^{\times})^2}\chi(ac(f_4(x,y)))\ |dxdy|,$$

where

$$f_1(x,y)=x^{12}(y_0+\mathfrak{p}^{j+1}y)^4((y_0-1+\mathfrak{p}^{j+1}y)^2+\mathfrak{p}^{8n}x^8(y_0+\mathfrak{p}^{j+1}y)^4),$$
$$f_2(x,y)=x^{12}(1+\mathfrak{p}^{j+1}y)^4(y^2+\mathfrak{p}^{8n-(2+2j)}x^8(1+\mathfrak{p}^{j+1}y)^4),$$
$$f_3(x,y)=x^{12}(1+\mathfrak{p}^{j+1}y)^4(y^2+x^8(1+\mathfrak{p}^{j+1}y)^4),$$

and

$$f_4(x,y)=x^{12}(1+\mathfrak{p}^{j+1}y)^4(\mathfrak{p}^{2+2j-8n}y^2+x^8(1+\mathfrak{p}^{j+1}y)^4).$$

We note that each $\overline{f}_i,(i=1,2,3,4)$, does not have singular points on $(\mathbb{F}_q^{\times})^2$, so we may use the change of variables (2.3.1) and proceed in a similar manner as in the computation of $Z(s,f,\chi,\Delta_3)$.

We want to call the attention of the reader to the fact that the definition of the f_i's above depends on the value of $|(\mathfrak{p}^{j+1}y)^2+\mathfrak{p}^{8n}x^8(1+\mathfrak{p}^{j+1}y)^4|$, which in turn depends on the explicit description of the set $\{(w,z)\in\mathbb{R}^2 \mid w\leq \min\{2z,8n\}\}$. The later set can be described explicitly by using the arithmetic Newton polygon of $f(x,y)=(y^3-x^2)^2+x^4y^4$, see Example 1 in Section 2.4.3.

Summarizing, when $\chi = \chi_{triv}$,

$$\begin{aligned} Z(s,f,\chi_{triv}) &= 2q^{-1}(1-q^{-1}) + \frac{q^{-2-4s}(1-q^{-1})}{(1-q^{-2-4s})} + \frac{q^{-7-16s}(1-q^{-1})^2}{(1-q^{-2-4s})(1-q^{-5-12s})} \\ &+ \frac{q^{-8-18s}(1-q^{-1})^2}{(1-q^{-3-6s})(1-q^{-5-12s})} + \frac{q^{-3-6s}(1-q^{-1})}{(1-q^{-3-6s})} + \frac{(1-q^{-1})^2 q^{-6-14s}}{(1-q^{-1-2s})(1-q^{-5-12s})} \\ &- \frac{(1-q^{-1})^2 q^{-9-20s}}{(1-q^{-1-2s})(1-q^{-9-20s})} + \frac{(q-2)(1-q^{-1})q^{-6-12s}}{(1-q^{-5-12s})} + \frac{(1-q^{-1})(q^{-10-20s})}{(1-q^{-9-20s})} \\ &+ \frac{q^{-9-20s}}{(1-q^{-1-s})(1-q^{-9-20s})}\{q^{-1}(q^{-1-s}-q^{-1})N + (1-q^{-1})^2(1-q^{-1-s}) \\ &- q^{-2}(1-q^{-1-s})T)\}, \end{aligned} \tag{2.3.4}$$

where $N = (q-1)\text{Card}\{x \in \mathbb{F}_q^\times \mid x^2 = -1\}$ and $T = \text{Card}\{(x,y) \in (\mathbb{F}_q^\times)^2 | y^2 + x^8 = 0\}$. When $\chi \neq \chi_{triv}$ and $\chi|_{1+\mathfrak{p}O_K} = \chi_{triv}$ we have several cases: If $\chi^2 = \chi_{triv}$, we have

$$Z(s,f,\chi) = \frac{(1-q^{-1})^2 q^{-6-14s}}{(1-q^{-1-2s})(1-q^{-5-12s})} - \frac{(1-q^{-1})^2 q^{-9-20s}}{(1-q^{-1-2s})(1-q^{-9-20s})}. \tag{2.3.5}$$

When $\chi^4 = \chi_{triv}$,

$$\begin{aligned} Z(s,f,\chi) = q^{-1}(1-q^{-1}) &+ \frac{q^{-3-4s}(1-q^{-1})}{(1-q^{-2-4s})} + \frac{q^{-2-4s}(1-q^{-1})^2}{(1-q^{-2-4s})} \\ &+ \frac{q^{-7-16s}(1-q^{-1})^2}{(1-q^{-2-4s})(1-q^{-5-12s})}. \end{aligned} \tag{2.3.6}$$

In the case where $\chi^6 = \chi_{triv}$, we obtain

$$\begin{aligned} Z(s,f,\chi) = \frac{q^{-8-18s}(1-q^{-1})^2}{(1-q^{-3-6s})(1-q^{-5-12s})} &+ \frac{q^{-3-6s}(1-q^{-1})^2}{(1-q^{-3-6s})} \\ &+ \frac{q^{-4-6s}(1-q^{-1})}{(1-q^{-3-6s})} + q^{-1}(1-q^{-1}). \end{aligned} \tag{2.3.7}$$

If $\chi^{12} = \chi_{triv}$, then

$$Z(s,f,\chi) = \overline{\chi}^4(\overline{y}_0)\overline{\chi}^2(\overline{y}_0 - 1)\frac{(q-2)(1-q^{-1})q^{-6-12s}}{(1-q^{-5-12s})}, \tag{2.3.8}$$

where $\bar{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}_q^\times$. Finally for $\chi^{20} = \chi_{triv}$

$$Z(s,f,\chi) = \frac{(1-q^{-1})(q^{-10-20s})}{(1-q^{-9-20s})}. \tag{2.3.9}$$

In all other cases $Z(s,f,\chi)=0$.

2.3.2 The local zeta function of $(y^3-x^2)^2(y^3-cx^2)+x^4y^4$

Let $g(x,y) = (y^3-x^2)^2(y^3-cx^2)+x^4y^4$, with $c \in O_K^\times$ and $c \not\equiv 1 \mod \mathfrak{p}$. In this example we assume that the characteristic of the residue field of K is different from 2 and 3. As in example 2.3.1, the origin of K is the only singular point of $g(x,y)$ and it is degenerate with respect to its geometric Newton polygon. The conical subdivision of $\mathbb{R}_+^2$ subordinated to the geometric Newton polygon of $g(x,y)$ is the same as in Table 1.1 and Figure 1.1.

Computation of $Z(s,g,\chi,\Delta_i), i=1,2,3,4,6,7,8,9.$

These integrals correspond to the case in which g is non–degenerate on Δ_i. The integral corresponding to Δ_6 can be calculated as follows.

$$\begin{aligned} Z(s,g,\chi,\Delta_6) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \int\limits_{\mathfrak{p}^{3n+2m}O_K^\times \times \mathfrak{p}^{2n+m}O_K^\times} \chi(ac\ g(x,y))\ |g(x,y)|^s |dxdy|, \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{(-5-18s)n+(-3-9s)m} \int\limits_{O_K^{\times 2}} \chi(ac\ g_6(x,y))|g_6(x,y)|^s\ |dxdy|, \end{aligned}$$

where $g_6(x,y) = (y^3-\mathfrak{p}^m x^2)^2(y^3-c\mathfrak{p}^m x^2)+\mathfrak{p}^{2n+3m}x^4y^4$, note that $\overline{g_6}(x,y)=y^9$. By using the change of variables (2.3.1) with the function g_6 and by applying Lemma 1.2.2, we obtain

$$Z(s,g,\chi,\Delta_6)=\begin{cases}\frac{q^{-8-27s}(1-q^{-1})^2}{(1-q^{-3-9s})(1-q^{-5-18s})} & \textit{if } \chi=\chi_{triv}\\ \\ \frac{q^{-8-27s}(1-q^{-1})^2}{(1-q^{-3-9s})(1-q^{-5-18s})} & \textit{if } \chi^9=\chi_{triv}, \chi|_U=\chi_{triv}\\ 0 & \textit{all other cases},\end{cases}$$

where $U=1+\mathfrak{p}O_K$.

We note here that for $i=1,2,3,4,7,8$ and 9, the computation of the $Z(s,f,\chi,\Delta_i)$ are similar to the case $Z(s,f,\chi,\Delta_6)$.

Computation of $Z(s,g,\chi,\Delta_5)$ (An integral on a degenerate face in the sense Kouchnirenko)

$$\begin{aligned}Z(s,g,\chi,\Delta_5) &= \sum_{n=1}^{\infty}\int\limits_{\mathfrak{p}^{3n}O_K^{\times}\times\mathfrak{p}^{2n}O_K^{\times}}\chi(ac\ g(x,y))|g(x,y)|^s|dxdy|,\\ &= \sum_{n=1}^{\infty}q^{-5n-18ns}\int\limits_{O_K^{\times 2}}\chi(ac(g^{(n)}(x,y))|g^{(n)}(x,y)|^s|dxdy|.\end{aligned}$$

where $g^{(n)}(x,y)=(y^3-x^2)^2(y^3-cx^2)+\mathfrak{p}^{2n}x^4y^4$, for $n\geqslant 1$. We use the map Φ defined in (2.3.3), giving $g^{(n)}\circ\Phi(x,y)=x^{18}y^6\widetilde{g^{(n)}}(x,y)$, with $\widetilde{g^{(n)}}(x,y)=(y-1)^2(y-c)+\mathfrak{p}^{2n}x^2y^2$, then we have to compute

$$\begin{aligned}I(s,g^{(n)},\chi) &:= \int\limits_{O_K^{\times 2}}\chi(ac(g^{(n)}(x,y))|g^{(n)}(x,y)|^s|dxdy|,\\ &= \int\limits_{O_K^{\times 2}}\chi(ac(x^{18}y^6\widetilde{g^{(n)}}(x,y)))|\widetilde{g^{(n)}}(x,y)|^s|dxdy|.\end{aligned}$$

We decompose $O_K^{\times 2}$ as follows:

$$O_K^{\times 2} = \left(O_K^{\times} \times \{y_0 + \mathfrak{p}O_K \mid y_0 \not\equiv 1, c \mod \mathfrak{p}\}\right) \cup \left(O_K^{\times} \times \{1 + \mathfrak{p}O_K\}\right) \\ \cup \left(O_K^{\times} \times \{c + \mathfrak{p}O_K\}\right),$$

where y_0 runs through a set of representatives of $\mathbb{F}_q^{\times}$ in O_K. By using the same strategy of example 2.3.1: we use an analytic bijection Φ over the units as a change of variables and then we split the integration domain according with the roots of the quasihomogeneous part of g. In each one of the sets of the splitting, calculations can be done by using the arithmetical non–degeneracy condition and/or the stationary phase formula. Thus we get

1. $\chi = \chi_{triv}$,

$$\begin{aligned} Z(s,f,\chi_{triv}) = 2q^{-1}(1-q^{-1}) &+ \frac{q^{-2-6s}(1-q^{-1})}{(1-q^{-2-6s})} + \frac{q^{-7-24s}(1-q^{-1})^2}{(1-q^{-2-6s})(1-q^{-5-18s})} \\ &+ \frac{q^{-8-27s}(1-q^{-1})^2}{(1-q^{-3-9s})(1-q^{-5-18s})} + \frac{q^{-3-9s}(1-q^{-1})}{(1-q^{-3-9s})} \\ &+ \frac{q^{-6-20s}U_0(q^{-s})}{(1-q^{-1-s})(1-q^{-6-20s})} + \frac{q^{-7-20s}(U_1(q^{-s})+(1-q^{-1})^2)}{(1-q^{-1-s})(1-q^{-7-20s})} \\ &+ \frac{(1-q^{-1})^2 q^{-6-20s}}{(1-q^{-1-2s})(1-q^{-5-18s})} - \frac{(1-q^{-1})^2 q^{-6-20s}}{(1-q^{-1-2s})(1-q^{-6-20s})} \\ &+ \frac{(1-q^{-1})^2(q^{-6-19s})}{(1-q^{-5-18s})(1-q^{-1-s})} + \frac{(q-3)(1-q^{-1})q^{-6-18s}}{(1-q^{-5-18s})} \\ &+ \frac{(1-q^{-1})(q^{-7-20s})}{(1-q^{-6-20s)}} - \frac{(1-q^{-1})(q^{-8-20s})}{(1-q^{-7-20s)}} \end{aligned} \tag{2.3.10}$$

where

$$U_0(q^{-s}) = q^{-2-s}(1-q^{-1})N_1 + T_2(1-q^{-1-s})\{(q-1)^2 - N_1\},$$
$$N_1 = \text{Card}\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^\times)^2 \mid \overline{a}^{18}(\overline{b}^2(1-\overline{c})+\overline{a}^2)=0\},$$
$$T_2 = \sum_{\substack{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}\\ (\overline{b}^2(1-\overline{c})+a^2)\neq 0}} \chi(ac(a^{18}(b^2(1-c)+a^2))),$$
$$U_1(q^{-s}) = q^{-2-s}(1-q^{-1})N_2 + T_3(1-q^{-1-s})\{(q-1)^2 - N_2\},$$
$$N_2 = \text{Card}\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^\times)^2 \mid \overline{a}^{18}\overline{b}\overline{c}^6(\overline{c}-1)^2+\overline{a}^{20}\overline{c}^2=0\},$$

and

$$T_3 = \sum_{\substack{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}\\ (\overline{b}^2(1-\overline{c})+\overline{a}^2)\neq 0}} \chi(ac(a^{18}(b^2(1-c)+a^2))).$$

2. $\chi^2 = \chi_{triv}$, and $\chi|_U = \chi_{triv}$, $U = 1 + \mathfrak{p}O_K$, we have

$$\begin{aligned} Z(s,f,\chi) = {} & \overline{\chi}(1-\overline{c})\frac{(1-q^{-1})^2 q^{-6-20s}}{(1-q^{-1-2s})(1-q^{-5-18s})} \\ & -\overline{\chi}(1-\overline{c})\frac{(1-q^{-1})^2 q^{-6-20s}}{(1-q^{-1-2s})(1-q^{-6-20s})} \\ & +\overline{\chi}(\overline{c}^6(\overline{c}-1)^2)\frac{(1-q^{-1})^2(q^{-6-19s})}{(1-q^{-5-18s})(1-q^{-1-s})} \\ & +\overline{\chi}(\overline{c}^6(\overline{c}-1)^2)\frac{(1-q^{-1})^2(q^{-7-20s})}{(1-q^{-7-20s})(1-q^{-1-s})}. \end{aligned} \tag{2.3.11}$$

where $\bar{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}_q^\times$.

3. $\chi^6 = \chi_{triv}$ and $\chi|_U = \chi_{triv}$,

$$\begin{aligned} & Z(s,f,\chi) = \\ & \bar{\chi}(-\overline{c})\left(q^{-1}(1-q^{-1}) + \frac{q^{-3-6s}(1-q^{-1}) + q^{-2-6s}(1-q^{-1})^2}{(1-q^{-2-6s})}\right) \\ & +\bar{\chi}(-\overline{c})\left(\frac{q^{-7-24s}(1-q^{-1})^2}{(1-q^{-2-6s})(1-q^{-5-18s})}\right), \end{aligned} \tag{2.3.12}$$

4. $\chi^9 = \chi_{triv}$ and $\chi|_U = \chi_{triv}$, we obtain

$$\begin{aligned} Z(s,f,\chi) = & \frac{q^{-8-27s}(1-q^{-1})^2}{(1-q^{-3-9s})(1-q^{-5-18s})} + \frac{q^{-3-9s}(1-q^{-1})^2}{(1-q^{-3-9s})} \\ & + \frac{q^{-4-9s}(1-q^{-1})}{(1-q^{-3-9s})} + q^{-1}(1-q^{-1}). \end{aligned} \tag{2.3.13}$$

5. $\chi^{18} = \chi_{triv}$ and $\chi|_U = \chi_{triv}$, then

$$Z(s,f,\chi) = \overline{\chi}(\overline{y_0}^7(\overline{y_0}-1))\frac{(q-3)(1-q^{-1})q^{-6-18s}}{(1-q^{-5-18s})}, \tag{2.3.14}$$

6. $\chi^{20} = \chi_{triv}$ and $\chi|_U = \chi_{triv}$

$$Z(s,f,\chi) = \frac{(1-q^{-1})(q^{-7-20s})}{(1-q^{-6-20s)}} - \overline{\chi}(\overline{c}^8)\frac{(1-q^{-1})(q^{-8-20s})}{(1-q^{-7-20s)}}. \tag{2.3.15}$$

7. In all other cases $Z(s,f,\chi) = 0$.

2.4 Integrals Over Degenerate Cones

From the examples in Section 2.3, we may deduce that when one deals with an integral of type $Z(s,f,\chi,\Delta)$ over a degenerate cone, we have to use an analytic bijection Φ over the units as a change of variables and then, split the integration domain according with the roots of the tangent cone of f. In each one of the sets of the splitting, calculations can be done by using the arithmetical non–degeneracy condition and/or the stationary phase formula. The purpose of this section is to show how this procedure works.

2.4.1 Some reductions on the integral $Z(s, f, \chi, \Delta)$

Proposition 2.4.1 ([27, Proposition 5.1]). *Let $f(x,y) \in O_K[x,y]$ be a semiquasihomogeneous polynomial, with respect to the weight (a,b), with a,b coprime, and*

$$f^{(m)}(x,y) := \mathfrak{p}^{-d_0 m} f(\mathfrak{p}^{am}x, \mathfrak{p}^{bm}y) = \sum_{j=0}^{l_f} \mathfrak{p}^{(d_j-d_0)m} f_j(x,y),$$

where $m \geqslant 1$, and

$$f_j(x,y) = c_j x^{u_j} y^{v_j} \prod_{i=1}^{l_j} (y^a - \alpha_{i,j} x^b)^{e_{i,j}}, c_j \in K^\times. \tag{2.4.1}$$

Then there exists a measure–preserving bijection

$$\begin{aligned} \Phi: \quad O_K^{\times 2} &\longrightarrow O_K^{\times 2} \\ (x,y) &\longmapsto (\Phi_1(x,y), \Phi_2(x,y)), \end{aligned}$$

such that $F^{(m)}(x,y) := f^{(m)} \circ \Phi(x,y) = x^{N_i} y^{M_i} \widetilde{f^{(m)}}(x,y)$, with

$$\widetilde{f^{(m)}}(x,y) = \sum_{j=0}^{l_f} \mathfrak{p}^{(d_j-d_0)m} \widetilde{f}_j(x,y),$$

where one can assume that $\widetilde{f}_j(x,y)$ is a polynomial of the form

$$\widetilde{f}_j(u,w) = c_j u^{A_j} w^{B_j} \prod_{i=1}^{l_j} (w - \alpha_{i,j})^{e_{i,j}}. \tag{2.4.2}$$

After using Φ as a change of variables in $Z(s, f, \chi, \Delta)$, one has to deal with integrals of type:

$$I(s, F^{(m)}, \chi) := \int_{O_K^{\times 2}} \chi(ac\ (F^{(m)}(x,y)))\ |F^{(m)}(x,y)|^s\ |dxdy|.$$

Integrals $I(s, F^{(m)}, \chi)$ will be computed in Propositions 2.4.2 and 2.4.3. The proof of these propositions are based on the corresponding Proposition in [18], but several simplifications were obtained. For the sake of completeness we present here the details of the proofs, also with the aim of introduced some notation that we will need in the remain of the chapter.

Proposition 2.4.2 ([27, Proposition 5.2])**.**

$$I(s, F^{(m)}, \chi) = \frac{U_0(q^{-s}, \chi)}{1 - q^{-1-s}} + \sum_{\{\theta \in O_K \mid f_0(1, \theta^a) = 0\}} J_\theta(s, m, \chi),$$

where $U_0(q^{-s}, \chi)$ is a polynomial with rational coefficients and

$$J_\theta(s, m, \chi) :=$$

$$\sum_{k=1+l(f_0)}^{\infty} q^{-k} \int_{O_K^{\times 2}} \chi(ac(F^{(m)}(x, \theta + \mathfrak{p}^k y)))\ |F^{(m)}(x, \theta + \mathfrak{p}^k y)|^s\ |dxdy|.$$

Proof. From Proposition 2.4.1

$$F^{(m)}(x, y) = x^{N_i} y^{M_i} \left(\sum_{j=0}^{l_f} \mathfrak{p}^{(d_j - d_0)m} \widetilde{f}_j(x, y) \right) = \sum_{j=0}^{l_f} \mathfrak{p}^{(d_j - d_0)m} f_j^*(x, y), \tag{2.4.3}$$

where

$$f_j^*(x, y) = c_j x^{A_j + N_i} y^{B_j + M_i} \prod_{i=1}^{l_j} (y - \alpha_{i,j})^{e_{i,j}}. \tag{2.4.4}$$

Set

$$R(f_0) := \{\theta \in O_K | f_0(1, \theta^a) = 0\}$$

$$l(f_0) := \max_{\substack{\theta \neq \theta' \\ \theta, \theta' \in R(f_0)}} \{v(\theta - \theta')\}, \text{ and}$$

$$B(\theta) = B(l(f_0), \theta) := O_K^\times \times \left(\theta + \mathfrak{p}^{1+l(f_0)} O_K\right),$$

for $\theta \in O_K$, with $v(\theta) \leqslant l(f_0)$. By subdividing $O_K^{\times 2}$ into equivalence classes modulo $\mathfrak{p}^{1+l(f_0)}$, we obtain that,

$$I(s, F^{(m)}, \chi) = \sum_{\theta \notin R(f_0)} \int_{B(\theta)} \chi(ac(F^{(m)}(x,y)))\ |F^m(x,y)|^s\ |dxdy|$$
$$+ \sum_{\theta \in R(f_0)} \int_{B(\theta)} \chi(ac(F^{(m)}(x,y)))\ |F^{(m)}(x,y)|^s\ |dxdy|.$$

Now we use the fact that $O_K = \sqcup_{k=0}^{\infty} (\mathfrak{p}^k O_K^\times)$ in $B(\theta)$. Thus $B(\theta) = O_K^\times \times (\theta + \mathfrak{p}^k O_K^\times)$, where $k \geq 1 + l(f_0)$ and our integral becomes

$$I(s, F^{(m)}, \chi) = \sum_{\theta \notin R(f_0)} \sum_{k=1+l(f_0)}^{\infty} q^{-k} \int_{B(\theta)} \chi(ac(F^{(m)}(x,y)))\ |F^m(x,y)|^s\ |dxdy|$$
$$+ \sum_{\theta \in R(f_0)} \sum_{k=1+l(f_0)}^{\infty} q^{-k} \int_{B(\theta)} \chi(ac(F^{(m)}(x,y)))\ |F^{(m)}(x,y)|^s\ |dxdy|. \qquad (2.4.5)$$

From (2.4.4), we have that for any $(x, y) \in O_K^{\times 2}$,

$$f_j^*(x, \theta + \mathfrak{p}^k y)$$
$$= \begin{cases} c_j x^{A_j+N_i} (\theta + \mathfrak{p}^k y)^{B_j+M_i} \prod_{i=1}^{l_j} \left((\theta - \alpha_{i,j}) + \mathfrak{p}^k y\right)^{e_{i,j}} & \text{if } f_j^*(1,\theta) \neq 0 \\ c_j x^{A_j+N_i} (\theta + \mathfrak{p}^k y)^{B_j+M_i} \prod_{\substack{i=1 \\ i \neq i_0}}^{l_j} \left((\theta - \alpha_{i,j}) + \mathfrak{p}^k y\right)^{e_{i,j}} \mathfrak{p}^{k e_{i_0,j}} y^{e_{i_0,j}} & \text{if } f_j^*(1,\theta) = 0, \end{cases}$$

where $\theta = \alpha_{i_0,j}$. We put

$$\gamma_j(x,y)$$
$$:= \begin{cases} x^{A_j+N_i}(\theta+\mathfrak{p}^k y)^{B_j+M_i}\prod_{i=1}^{l_j}\left((\theta-\alpha_{i,j})+\mathfrak{p}^k y\right)^{e_{i,j}} & \text{if } f_j^*(1,\theta)\neq 0 \\ x^{A_j+N_i}(\theta+\mathfrak{p}^k y)^{B_j+M_i}\prod_{\substack{i=1\\ i\neq i_0}}^{l_j}\left((\theta-\alpha_{i,j})+\mathfrak{p}^k y\right)^{e_{i,j}}\mathfrak{p}^{ke_{i_0,j}}y^{e_{i_0,j}} & \text{if } f_j^*(1,\theta)=0, \end{cases}$$

and note that in both cases the γ_j are polynomials satisfying $|\gamma_j(x,y)| = 1$, for any $(x,y)\in O_K^{\times 2}$. By abuse of notation we will write

$$f_j^*(x,\theta+\mathfrak{p}^k y) = c_j\gamma_j(x,y)\mathfrak{p}^{ke_{\theta,j}}y^{e_{\theta,j}}. \tag{2.4.6}$$

Finally we return to the computation of the integral $I(s,F^{(m)},\chi)$. Note that if $\theta\notin R(f_0)$ then from (2.4.3) and (2.4.6) we get that $\overline{F^{(m)}(x,\theta+\mathfrak{p}^k y)}$ has no singular points over $(\mathbb{F}_q^{\times})^2$, therefore we may apply Lemma 1.2.3 in 2.4.5 to obtain the desired conclusion. □

The next step is to compute the integral $J_\theta(s,m,\chi)$, we introduce here some notation. For a polynomial $h(x,y)\in O_K[x,y]$ we define $N_h = \text{Card}\{(\overline{x}_0,\overline{y}_0) \in (\mathbb{F}_q{}^{\times})^2 \mid \overline{h}(\overline{x}_0,\overline{y}_0)=0\}$, and put

$$M_h = \frac{q^{-s}(1-q^{-1})N_h}{1-q^{-1-s}} + (q-1)^2 - N_h \quad \text{and} \quad \Sigma_h := \sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q{}^{\times})^2\\ \overline{h}(\overline{a},\overline{b})\neq 0}} \chi(ac\ (h(a,b))).$$

Proposition 2.4.3. *We fix $\theta\in R(f_0)$ and assume that $f(x,y)$ is arithmetically non degenerate with respect to $\Gamma_{f,\theta}$ (see Definition 2.1.2) . Let $\tau_i, i = 0,1,2,\cdots,r$ be the abscissas of the vertices of $\Gamma_{f,\alpha_{i,0}}$, cf.* (2.4.2) *and Definition 2.1.1.*

1. *$J_\theta(s,m,\chi_{triv})$ is equal to*

$$\sum_{i=0}^{r-1} q^{-(D_{i+1}-d_0)ms}\left(\frac{q^{-(1+s\varepsilon_{i+1})([m\tau_i]+1)}-q^{-(1+s\varepsilon_{i+1})([m\tau_{i+1}]-1)}}{1-q^{-(1+s\varepsilon_{i+1})}}\right)M_g$$
$$+q^{-(D_{r+1}-d_0)ms}\left(\frac{q^{-(1+s\varepsilon_{r+1})[m\tau_r]}}{1-q^{-(1+s\varepsilon_{r+1})}}\right)M_{g_r}+\sum_{i=1}^{r} q^{-(D_i-d_0)ms-(s\varepsilon_i[m\tau_i])}M_G,$$

with

$$g(x,y)=\gamma_{i+1}(x,y)y^{e_{i+1,\theta}}+\mathfrak{p}^{m(D_{i+1}-D_i)}(\textit{higher order terms}),$$
$$g_r(x,y)=\gamma_{r+1}(x,y)y^{e_{r+1,\theta}}+\mathfrak{p}^{m(D_{r+1}-D_i)}(\textit{higher order terms}),$$

and

$$G(x,y)=\sum_{\widetilde{w}_{i,\theta}(\mathcal{V}_i)=0}\gamma_i(x,y)y^{e_{i,\theta}},$$

where $\widetilde{w}_{i,\theta}(\widetilde{z})$ is the straight line corresponding to the term

$$\mathfrak{p}^{(d_j-d_0)m+ke_{j,\theta}}\gamma_j(x,y)y^{e_{j,\theta}},$$

cf. (2.1.5)*.*

2. *In the case $\chi|_{1+\mathfrak{p}O_K}=\chi_{triv}$, $J_\theta(s,m,\chi)$ is equal to*

$$\sum_{i=0}^{r-1} q^{-(D_{i+1}-d_0)ms}\left(\frac{q^{-(1+s\varepsilon_{i+1})([m\tau_i]+1)}-q^{-(1+s\varepsilon_{i+1})([m\tau_{i+1}]-1)}}{1-q^{-(1+s\varepsilon_{i+1})}}\right)\Sigma_g$$
$$+q^{-(D_{r+1}-d_0)ms}\left(\frac{q^{-(1+s\varepsilon_{r+1})[m\tau_r]}}{1-q^{-(1+s\varepsilon_{r+1})}}\right)\Sigma_{g_r}+\sum_{i=1}^{r} q^{-(D_i-d_0)ms-(s\varepsilon_i[m\tau_i])}.$$

3. *In all other cases $J_\theta(s,m,\chi)=0$.*

Proof. From and (2.4.3) and (2.4.6) we have

$$F^{(m)}(x,\theta+\mathfrak{p}^k y)=\sum_{j=0}^{l_f} c_j \mathfrak{p}^{(d_j-d_0)m+ke_{j,\theta}}\gamma_j(x,y)y^{e_{j,\theta}}. \tag{2.4.7}$$

Then we associate to each term in (2.4.7) a straight line of the form $\widetilde{w}_{j,\theta}(\widetilde{z}) := (d_j - d_0)m + e_{j,\theta}\widetilde{z}$, for $j = 0, 1, \ldots, l_f$. We also associate to $F^{(m)}(x, \theta + \mathfrak{p}^k y)$ the convex set

$$\Gamma_{F^{(m)}(x,\theta+\mathfrak{p}^k y)} = \{(\widetilde{z},\widetilde{w}) \in \mathbb{R}^2_+ \mid \widetilde{w} \leqslant \min_{0\leqslant j\leqslant l_f}\{\widetilde{w}_{j,\theta}(\widetilde{z})\}\}.$$

As it was noticed in [27], the polygon $\Gamma_{F^{(m)}(x,\theta+\mathfrak{p}^k y)}$ is a rescaled version of $\Gamma_{f,\theta}$. Thus the vertices of $\Gamma_{F^{(m)}(x,\theta+\mathfrak{p}^k y)}$ can be described in terms of the vertices of $\Gamma_{f,\theta}$. More precisely, the vertices of $\Gamma_{F^{(m)}(x,\theta+\mathfrak{p}^k y)}$ are

$$\mathcal{V}_i := \begin{cases} (0,0) & \textit{if } i = 0 \\ (m\tau_i, (D_i - d_0)m + m\varepsilon_i\tau_i) & \textit{if } i = 1, 2, \ldots, r, \end{cases}$$

where the τ_i are the abscissas of the vertices of $\Gamma_{f^{(m)},\theta}$. The crucial fact in our proof is that $F^{(m)}(x, \theta + \mathfrak{p}^k y)$, may take different forms depending of the place that k occupies with respect to the abscissas of the vertices of $\Gamma_{F^{(m)}(x,\theta+\mathfrak{p}^k y)}$. This leads to the cases: (i) $m\tau_i < k < m\tau_{i+1}$, (ii) $k > m\tau_r$, and (iii) $k = m\tau_i$.

Case (i): $\mathbf{m\tau_i < k < m\tau_{i+1}}$. There exists some $j_l \in \{0, \ldots, l_f\}$ such that

$$(d_{j_l} - d_0)m + k\varepsilon_{j_l} = (\mathcal{D}_{i+1} - d_0)m + k\varepsilon_{i+1},$$

and

$$(d_{j_l} - d_0)m + k\varepsilon_{j_l} < (d_j - d_0)m + k\varepsilon_j,$$

for $j \in \{0, \ldots, l_f\} \setminus \{j_l\}$. In consequence

$$F^{(m)}(x,\theta+\mathfrak{p}^k y) = \mathfrak{p}^{-(D_{i+1}-d_0)m-\varepsilon_{i+1}k}(\gamma_{i+1}(x,y)y^{e_{i+1,\theta}} + \mathfrak{p}^{m(D_{i+1}-D_i)}(\cdots))$$

for any $(x, y) \in O_K^{\times 2}$, where

$$\gamma_{i+1}(x,y)y^{e_{i+1,\theta}} + \mathfrak{p}^{m(D_{i+1}-D_i)}(\cdots)$$
$$= \gamma_{i+1}(x,y)y^{e_{i+1,\theta}} + \mathfrak{p}^{m(D_{i+1}-D_i)}(\text{terms with weighted degree} \geqslant D_{i+1}).$$

We put $g(x,y) := \gamma_{i+1}(x,y)y^{e_{i+1,\theta}} + \mathfrak{p}^{m(D_{i+1}-D_i)}(\cdots)$. Then

$$\int_{O_K^{\times 2}} \chi(ac(F^{(m)}(x, \theta + \mathfrak{p}^k y)))\ |F^{(m)}(x, \theta + \mathfrak{p}^k y)|^s\ |dxdy|$$
$$= q^{-(D_{i+1}-d_0)ms - \varepsilon_{i+1}ks} \int_{O_K^{\times 2}} \chi(ac(g(x,y))\ |g(x,y)|^s\ |dxdy|.$$

By using the following partition of $O_K^{\times^2}$,

$$O_K^{\times^2} = \bigsqcup_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} (a,b) + (\mathfrak{p}O_K)^2, \tag{2.4.8}$$

we have

$$\int_{O_K^{\times 2}} \chi(ac(g(x,y))\ |g(x,y)|^s\ |dxdy|$$
$$= \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int_{(a,b)+(\mathfrak{p}O_K)^2} \chi(ac\ g(x,y))|g(x,y)|^s|dxdy| \tag{2.4.9}$$
$$= \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int_{O_K^2} \chi(ac\ g(a+\mathfrak{p}x, b+\mathfrak{p}y))\ |g(a+\mathfrak{p}x, b+\mathfrak{p}y)|^s\ |dxdy|.$$

By definition of $\gamma_j(x,y)$ (in proof of Proposition 1.5.2), we see that $\frac{\partial \overline{g}}{\partial y}(x,y) = e_{i+1,\theta}y^{e_{i+1,\theta}-1}$ then $\frac{\partial \overline{g}}{\partial y}(\overline{a},\overline{b}) \not\equiv 0(mod\ \mathfrak{p})$ for $(\overline{a},\overline{b}) \in (\mathbb{F}_q^\times)^2$. Therefore the following is a measure preserving map from O_K^2 to itself:

$$\begin{cases} z_1 = x \\ z_2 = \frac{g(a+\mathfrak{p}x, b+\mathfrak{p}y) - g(a,b)}{\mathfrak{p}}. \end{cases} \tag{2.4.10}$$

By using (2.4.10) as a change of variables, (2.4.9) becomes:

$$\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K} \chi(ac\ (g(a,b)+\mathfrak{p}z_2))\ |g(a,b)+\mathfrak{p}z_2|^s\ |dz_2|,$$

and then Lemma 1.2.2 implies that the later sum equals

$$\begin{cases} \frac{q^{-s}(1-q^{-1})N_g}{(1-q^{-1-s})} + (q-1)^2 - N_g & \textit{if } \chi = \chi_{triv} \\ \\ \sum\limits_{\substack{(\overline{a},\overline{b})\in({\mathbb{F}_q}^{\times})^2 \\ \overline{g}(\overline{a},\overline{b})\neq 0}} \chi(ac(g(a,b))) & \textit{if } \ \chi|_U = \chi_{triv} \\ 0 & \textit{all other cases}, \end{cases}$$

where $U = 1 + \mathfrak{p}O_K$, and $N_g = \mathrm{Card}\{(\overline{a},\overline{b}) \in ({\mathbb{F}_q}^{\times})^2 \mid \overline{g}(\overline{a},\overline{b}) = 0\}$.

Case (ii): $\mathbf{k > m\tau_r}$. There exists some $j_p \in \{0,\dots,l_f\}$ such that $(d_{j_p} - d_0)m + k\varepsilon_{j_p} = (\mathcal{D}_{r+1} - d_0)m + k\varepsilon_{r+1}$, and $(d_{j_p} - d_0)m + k\varepsilon_{j_p} < (d_j - d_0)m + k\varepsilon_j$, for $j \in \{0,\dots,l_f\} \setminus \{j_p\}$. Therefore

$$F^{(m)}(x, \theta + \mathfrak{p}^k y) = \mathfrak{p}^{-(D_{r+1}-d_0)m-\varepsilon_{r+1}k}(\gamma_{r+1}(x,y)y^{e_{r+1,\theta}} + \mathfrak{p}^{m(D_{r+1}-D_i)}(\cdots))$$

for any $(x,y)\in O_K^{\times 2}$. A similar reasoning as in the previous case, shows that

$$\int\limits_{O_K^{\times 2}} \chi(ac(F^{(m)}(x,\theta+\mathfrak{p}^k y)))\ |F^{(m)}(x,\theta+\mathfrak{p}^k y)|^s\ |dxdy|$$

$$=\begin{cases} \frac{q^{-(D_{r+1}-d_0)ms-\varepsilon_{r+1}ks}q^{-s}(1-q^{-1})N_r}{(1-q^{-1-s})}+(q-1)^2-N_r & \textit{if } \chi=\chi_{triv} \\ q^{-(D_{r+1}-d_0)ms-\varepsilon_{r+1}ks}\sum\limits_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q{}^{\times})^2 \\ \overline{g_r}(\overline{a},\overline{b})\neq 0}}\chi(ac(g_r(a,b))) & \textit{if } \ \chi|_U=\chi_{triv} \\ 0 & \textit{all other cases.}\end{cases}$$

Here

$$g_r(x,y)=\gamma_{r+1}(x,y)y^{e_{r+1,\theta}}+\mathfrak{p}^{m(D_{r+1}-D_i)}(\cdots)$$

and

$$N_r=\text{Card}\{(\overline{a},\overline{b})\in(\mathbb{F}_q{}^{\times})^2 \mid \overline{g_r}(\overline{a},\overline{b})=0\}.$$

Case (iii): $\mathbf{k}=\mathbf{m}\tau_{\mathbf{i}}$. There are some $j's\in\{0,\ldots,l_f\}$ such that

$$(d_{j_1}-d_0)m+k\varepsilon_{j_1}=\cdots=(d_{j_t}-d_0)m+k\varepsilon_{j_t}=(\mathcal{D}_i-d_0)m+k\varepsilon_i,$$

and for the remaining j's,

$$(\mathcal{D}_i-d_0)m+k\varepsilon_i<(d_j-d_0)m+k\varepsilon_j.$$

In this case

$$F^{(m)}(x,\theta+\mathfrak{p}^k y)=\mathfrak{p}^{-(D_i-d_0)m-\varepsilon_i k}(F^{(m)}_{\mathcal{V}_i}(x,y)+\mathfrak{p}^{m(D_{i+1}-D_i)}(\cdots))$$

for any $(x,y) \in O_K^{\times 2}$, where

$$F_{\mathcal{V}_i}^{(m)}(x,y) = \sum_{\widetilde{w}_{i,\theta}(\mathcal{V}_i)=0} \gamma_i(x,y) y^{e_{i,\theta}},$$

and $\widetilde{w}_{i,\theta}(\widetilde{z})$ is the straight line corresponding to the term $\mathfrak{p}^{(d_j-d_0)m+ke_{j,\theta}}\gamma_j(x,y)y^{e_{j,\theta}}$. Therefore

$$\begin{aligned}&\int_{O_K^{\times 2}} \chi(ac(F^{(m)}(x,\theta+\mathfrak{p}^k y)))\ |F^{(m)}(x,\theta+\mathfrak{p}^k y)|^s\ |dxdy| \\ &= q^{-(D_i-d_0)ms-\varepsilon_i ks} \int_{O_K^{\times 2}} \chi(ac(G(x,y))|G(x,y)|^s\ |dxdy|,\end{aligned}$$

where $G(x,y) = F_{\mathcal{V}_i}^{(m)}(x,y) + \mathfrak{p}^{m(D_{i+1}-D_i)}(\cdots)$, then the arithmetical non degeneracy condition over f implies that some partial derivative of $\overline{G}$ is different from zero mod $\mathfrak{p}$, lets say $\frac{\partial \overline{G}}{\partial y}(\overline{a},\overline{b}) \not\equiv 0 \mod \mathfrak{p}$ for $(\overline{a},\overline{b}) \in (\mathbb{F}_q^\times)^2$. So we may use the same strategy as in case (i), to obtain

$$\int_{O_K^{\times 2}} \chi(ac(F^{(m)}(x,\theta+\mathfrak{p}^k y)))\ |F^{(m)}(x,\theta+\mathfrak{p}^k y)|^s\ |dxdy|$$

$$= \begin{cases} \frac{q^{-(D_i-d_0)ms-\varepsilon_i ks} q^{-s}(1-q^{-1})N_G}{(1-q^{-1-s})} + (q-1)^2 - N_G & \textit{if } \chi = \chi_{triv} \\ \\ q^{-(D_i-d_0)ms-\varepsilon_i ks} \sum\limits_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q{}^\times)^2 \\ \overline{G}(\overline{a},\overline{b})\neq 0}} \chi(ac(G(a,b))) & \textit{if } \ \chi|_U = \chi_{triv} \\ \\ 0 & \textit{all other cases}, \end{cases}$$

where $N_G = \operatorname{Card}\{(\overline{a},\overline{b}) \in (\mathbb{F}_q{}^\times)^2 \mid \overline{G}(\overline{a},\overline{b}) = 0\}$.

At this point we note that any $k \in \mathbb{N}, k \geqslant 1$, satisfies only one of the following conditions:

$$\begin{cases} [m\tau_i] \leqslant k \leqslant [m\tau_{i+1}] - 1, & \text{for } i = 0, 1, \ldots, r-1, \\ k = [m\tau_i], & \text{for } i = 0, 1, \ldots, r, \\ k \geqslant [m\tau_r] + 1, & \end{cases}$$

where $[x]$ denotes the greatest integer less than or equal to $x \in \mathbb{R}$.

Finally, from cases $(i), (ii), (iii)$ and the previous observation, we have that

$$J_\theta(s, m, \chi_{triv}) =$$
$$\sum_{k=1+l(f_0)}^{\infty} q^{-k} \int_{O_K^{\times 2}} \chi(ac(F^{(m)}(x, \theta + \mathfrak{p}^k y)))\ |F^{(m)}(x, \theta + \mathfrak{p}^k y)|^s\ |dxdy|$$
$$= \sum_{i=0}^{r-1} q^{-(D_{i+1}-d_0)ms} \sum_{k=[m\tau_i]+1}^{[m\tau_{i+1}]-1} q^{-k(1+s\varepsilon_{i+1})} M_g$$
$$+ q^{-(D_{r+1}-d_0)ms} \sum_{k=[m\tau_r]+1}^{\infty} q^{-k(1+s\varepsilon_{r+1})} M_{g_r} + \sum_{i=1}^{r} q^{-(D_i-d_0)ms-(s\varepsilon_i[m\tau_i])} M_G.$$

Some of the sums appearing in the previous expression can be estimated by means of the following algebraic identity $\sum_{k=A}^{B} z^k = \frac{z^A - z^{B+1}}{1-z}$. We get

$$J_\theta(s, m, \chi_{triv})$$
$$= \sum_{i=0}^{r-1} q^{-(D_{i+1}-d_0)ms} \left(\frac{q^{-(1+s\varepsilon_{i+1})([m\tau_i]+1)} - q^{-(1+s\varepsilon_{i+1})([m\tau_{i+1}]-1)}}{1 - q^{-(1+s\varepsilon_{i+1})}} \right) M_g$$
$$+ q^{-(D_{r+1}-d_0)ms} \left(\frac{q^{-(1+s\varepsilon_{r+1})[m\tau_r]}}{1 - q^{-(1+s\varepsilon_{r+1})}} \right) M_{g_r} + \sum_{i=1}^{r} q^{-(D_i-d_0)ms-(s\varepsilon_i[m\tau_i])} M_G.$$

Finally, when $\chi|_U = \chi_{triv}$, we have

$$\begin{aligned} &J_\theta(s,m,\chi) \\ &= \sum_{i=0}^{r-1} q^{-(D_{i+1}-d_0)ms}\left(\frac{q^{-(1+s\varepsilon_{i+1})([m\tau_i]+1)} - q^{-(1+s\varepsilon_{i+1})([m\tau_{i+1}]-1)}}{1-q^{-(1+s\varepsilon_{i+1})}}\right)\Sigma_g \\ &+q^{-(D_{r+1}-d_0)ms}\left(\frac{q^{-(1+s\varepsilon_{r+1})([m\tau_r]+1)}}{1-q^{-(1+s\varepsilon_{r+1})}}\right)\Sigma_{g_r} \\ &+\sum_{i=1}^{r} q^{-(D_i-d_0)ms-(-1-s\varepsilon_i[m\tau_i])}\Sigma_G. \end{aligned}$$

□

2.4.2 Poles of $Z(s,f,\chi,\Delta)$

Definition 2.4.1. *For a semi quasihomogeneous polynomial* $f(x,y) \in K[x,y]$ *which is arithmetically non degenerate with respect to*

$$\Gamma^A(f) = \bigcup_{\{\theta\in O_K | f_0(1,\theta^a)=0\}} \Gamma_{f,\theta},$$

we define

$$\mathcal{P}(\Gamma_{f,\theta}) := \bigcup_{i=1}^{r_\theta}\left\{-\frac{1}{\varepsilon_i}, -\frac{(a+b)+\tau_i}{\mathcal{D}_{i+1}+\varepsilon_{i+1}\tau_i}, -\frac{(a+b)+\tau_i}{\mathcal{D}_i+\varepsilon_i\tau_i}\right\} \cup \bigcup_{\{\varepsilon_{r+1}\neq 0\}}\left\{-\frac{1}{\varepsilon_{r+1}}\right\},$$

and

$$\mathcal{P}(\Gamma^A(f)) := \bigcup_{\{\theta\in O_K | f_0(1,\theta^a)=0\}} \mathcal{P}(\Gamma_{f,\theta}).$$

Where $\mathcal{D}_i, \varepsilon_i, \tau_i$ *are obtained form the equations of the straight segments that form the boundary of* $\Gamma_{f,\theta}$, *cf.* (2.1.2),(2.1.3), *and* (2.1.4).

Theorem 2.4.1. *Let* $f(x,y) = \sum_{j=0}^{l_f} f_j(x,y) \in O_K[x,y]$ *be a semi- quasihomogeneous* *polynomial, with respect to the weight* (a,b), *with* a,b *coprime, and* $f_j(x,y)$ *as in (2.4.1). If* $f(x,y)$ *is arithmetically non–degenerate with respect to* $\Gamma^A(f)$, *then the real parts of the poles of* $Z(s,f,\chi,\Delta)$ *belong to the set*

$$\{-1\} \cup \left\{-\frac{a+b}{d_0}\right\} \cup \{\mathcal{P}(\Gamma^A(f))\}.$$

In addition, $Z(s,f,\chi,\Delta)=0$ *for almost all* χ. *More precisely, if* $\chi|_{1+\mathfrak{p}O_K} \neq \chi_{triv}$, $Z(s,f,\chi,\Delta)=0$.

Proof. Let $\Delta := (a,b)\mathbb{R}_+$, then the integral $Z(s,f,\chi,\Delta)$ admits the following expansion:

$$\begin{aligned} Z(s,f,\chi,\Delta) &= \sum_{m=1}^{\infty} \int\limits_{\mathfrak{p}^{am}O_K^{\times}\times\mathfrak{p}^{bm}O_K^{\times}} \chi(ac(f(x,y))\ |f(x,y)|^s\ |dxdy| \\ &= \sum_{m=1}^{\infty} q^{-(a+b)m-d_0ms} \int\limits_{O_K^{\times 2}} \chi(ac\ (F^{(m)}(x,y)))\ |F^{(m)}(x,y)|^s\ |dxdy|, \end{aligned} \tag{2.4.11}$$

cf. 2.4.3 and cf. 2.4.7. From Proposition 2.4.2,

$$\int\limits_{O_K^{\times 2}} \chi(ac\ (F^{(m)}(x,y)))\ |F^{(m)}(x,y)|^s\ |dxdy| = \frac{U_0(q^{-s},\chi)}{1-q^{-1-s}} + \sum_{\{\theta\in O_K | f_0(1,\theta^a)=0\}} J_\theta(s,m,\chi),$$

thus (2.4.11) implies

$$Z(s,f,\chi,\Delta) = \frac{U_0(q^{-s},\chi)}{1-q^{-1-s}} + \sum_{\{\theta\in O_K^{\times} | f_0(1,\theta^a)=0\}} \left(\sum_{m=1}^{\infty} q^{-(a+b)m-d_0ms} J_\theta(s,m,\chi)\right).$$

Next we use the explicit formula for $J_\theta(s,m,\chi)$ given in Proposition 2.4.3 to obtain

$$\sum_{m=1}^{\infty} q^{-(a+b)m-d_0ms} J_\theta(s,m,\chi_{triv}) \tag{2.4.12}$$
$$= \sum_{i=0}^{r-1}\sum_{m=1}^{\infty} \frac{q^{-(a+b)m-([m\tau_i]+1)-(D_{i+1}m+\varepsilon_{i+1}([m\tau_i]+1))s}}{1-q^{-(1+s\varepsilon_{i+1})}} M_g$$
$$-\sum_{i=0}^{r-1}\sum_{m=1}^{\infty} \frac{q^{-(a+b)m-([m\tau_{i+1}]-1)-(D_{i+1}m+\varepsilon_{i+1}([m\tau_{i+1}]-1))s}}{1-q^{-(1+s\varepsilon_{i+1})}} M_g$$
$$+\sum_{m=1}^{\infty} \frac{q^{-(a+b)m-([m\tau_r]+1)-(D_{r+1}m+\varepsilon_{r+1}([m\tau_r]+1))s}}{1-q^{-(1+s\varepsilon_{r+1})}} M_{g_r}$$
$$+\sum_{i=1}^{r}\sum_{m=1}^{\infty} q^{(a+b)m-[m\tau_i]-(D_im-\varepsilon_i[m\tau_i])s} M_G.$$

Remark 2.4.1. *In order to compute the expression for the integral $J_\theta(s,m,\chi_{triv})$ we have to estimate sums of type*

$$\sum_{m=1}^{\infty} q^{-[m\tau_i]}.$$

Recall that $\tau_i = \frac{D_{i+1}-D_i}{\varepsilon_i-\varepsilon_{i+1}}$. Assume that $m = n(\varepsilon_i-\varepsilon_{i+1})+l$, where $l \in \{0,\cdots,\varepsilon_i-\varepsilon_{i+1}-1\}$, and $n \in \mathbb{N}\setminus\{0\}$. Then

$$[m\tau_i] = n(D_{i+1}-D_i) + [l\tau_i].$$

Therefore $\sum_{m=1}^{\infty} q^{-[m\tau_i]} = \sum_{l=0}^{\varepsilon_i-\varepsilon_{i+1}-1} \sum_{n \geqslant \frac{1-l}{(\varepsilon_i-\varepsilon_{i+1})}} q^{-n(D_{i+1}-D_i)+[l\tau_i]}$.

Now we go back to the computation of $J_\theta(s,m,\chi_{triv})$, from (2.4.12)

$$\sum_{m=1}^{\infty} q^{-(a+b)m-d_0ms} J_\theta(s,m,\chi_{triv}) \tag{2.4.13}$$

$$= \sum_{i=0}^{r-1}\Bigg\{\frac{1}{1-q^{-1-s\varepsilon_{i+1}}}\sum_{l=0}^{\varepsilon_i-\varepsilon_{i+1}-1}\sum_{n\geqslant\frac{1-l}{\varepsilon_i-\varepsilon_{i+1}}}\Big\{q^{-(a+b)l-[l\tau_i]-1-\{D_{i+1}l+\varepsilon_{i+1}[l\tau_i]+\varepsilon_{i+1}\}s}$$

$$q^{-n\{(a+b)(\varepsilon_i-\varepsilon_{i+1})+(D_{i+1}-D_i)-\{D_{i+1}(\varepsilon_i-\varepsilon_{i+1})-\varepsilon_{i+1}(D_{i+1}-D_i)\}s\}}M_g\Big\}\Bigg\}$$

$$-\sum_{i=0}^{r-1}\Bigg\{\frac{1}{1-q^{-1-s\varepsilon_{i+1}}}\sum_{l=0}^{\varepsilon_{i+1}-\varepsilon_{i+2}-1}\sum_{n\geqslant\frac{1-l}{\varepsilon_{i+1}-\varepsilon_{i+2}}}\Big\{q^{-1-(a+b)l+[l\tau_{i+1}]-\{D_{i+1}l-\varepsilon_{i+1}[l\tau_{i+1}]-\varepsilon_{i+1}\}s}$$

$$q^{-n\{(a+b)(\varepsilon_{i+1}-\varepsilon_{i+2})+(D_{i+2}-D_{i+1})+\{\varepsilon_{i+1}(D_{i+2}-D_{i+1})+D_{i+1}(\varepsilon_{i+1}-\varepsilon_{i+2})\}s\}}M_g\Big\}\Bigg\}$$

$$+\frac{1}{1-q^{-1-s\varepsilon_{r+1}}}\sum_{l=0}^{\varepsilon_i-\varepsilon_{i+1}-1}\sum_{n\geqslant\frac{1-l}{\varepsilon_i-\varepsilon_{i+1}}}\Big\{q^{-1-(a+b)l+[l\tau_{r+1}]-\{D_{r+1}l-\varepsilon_{r+1}[l\tau_{r+1}]-\varepsilon_{r+1}\}s}$$

$$q^{-n\{(a+b)(\varepsilon_r-\varepsilon_{r+1})+(D_{r+1}-D_r)+\{\varepsilon_{r+1}(D_{r+1}-D_r)+D_{r+1}(\varepsilon_r-\varepsilon_{r+1})\}s\}}M_{g_r}\Big\}$$

$$+\sum_{i=1}^{r}\Bigg\{\sum_{l=0}^{\varepsilon_i-\varepsilon_{i+1}-1}\sum_{n\geqslant\frac{1-l}{\varepsilon_i-\varepsilon_{i+1}}}\Big\{q^{-(a+b)l-[l\tau_i]-\{D_il-\varepsilon_i[l\tau_i]\}s}$$

$$q^{-n\{(a+b)(\varepsilon_i-\varepsilon_{i+1})+(D_{i+1}-D_i)+\{\varepsilon_i(D_{i+1}-D_i)+D_i(\varepsilon_i-\varepsilon_{i+1})\}s\}}M_G\Big\}\Bigg\}.$$

Next we compute the geometric series appearing in the latter expression, this gives

$$
\begin{aligned}
&\sum_{m=1}^{\infty} q^{-(a+b)m-d_0 ms} J_\theta(s,m,\chi_{triv}) \\
&= \sum_{i=0}^{r-1}\left\{ \frac{1}{1-q^{-1-s\varepsilon_{i+1}}} \left\{ \frac{q^{-1-\varepsilon_{i+1}s-(a+b)(\varepsilon_i-\varepsilon_{i+1})-(D_{i+1}-D_i)-\{\varepsilon_{i+1}(D_{i+1}-D_i)-D_{i+1}(\varepsilon_i-\varepsilon_{i+1})\}s}}{1-q^{-\{(a+b)(\varepsilon_i-\varepsilon_{i+1})+(D_{i+1}-D_i)+\{\varepsilon_{i+1}(D_{i+1}-D_i)+D_{i+1}(\varepsilon_i-\varepsilon_{i+1})\}s\}}} \right.\right. \\
&\quad \left.\left. + \sum_{l=1}^{\varepsilon_i-\varepsilon_{i+1}-1} \frac{q^{-(a+b)l-[l\tau_i]-1-\{D_{i+1}l+\varepsilon_{i+1}[l\tau_i]+\varepsilon_{i+1}\}s}}{1-q^{-\{(a+b)(\varepsilon_i-\varepsilon_{i+1})+(D_{i+1}-D_i)+\{\varepsilon_{i+1}(D_{i+1}-D_i)+D_{i+1}(\varepsilon_i-\varepsilon_{i+1})\}s\}}} \right\} M_g \right\} \\
&- \sum_{i=0}^{r-1}\left\{ \frac{1}{1-q^{-1-s\varepsilon_{i+1}}} \left\{ \frac{q^{-1-\varepsilon_{i+1}s-(a+b)(\varepsilon_{i+1}-\varepsilon_{i+2})-(D_{i+2}-D_{i+1})-\{\varepsilon_{i+1}(D_{i+2}-D_{i+1})-D_{i+1}(\varepsilon_{i+1}-\varepsilon_{i+2})\}s}}{1-q^{-\{(a+b)(\varepsilon_{i+1}-\varepsilon_{i+2})+(D_{i+2}-D_{i+1})+\{\varepsilon_{i+1}(D_{i+2}-D_{i+1})+D_{i+1}(\varepsilon_{i+1}-\varepsilon_{i+2})\}s\}}} \right.\right. \\
&\quad \left.\left. + \sum_{l=1}^{\varepsilon_{i+1}-\varepsilon_{i+2}-1} \frac{q^{-(a+b)l-[l\tau_{i+1}]-1-\{D_{i+1}l+\varepsilon_{i+1}[l\tau_{i+1}]+\varepsilon_{i+1}\}s}}{1-q^{-\{(a+b)(\varepsilon_{i+1}-\varepsilon_{i+2})+(D_{i+2}-D_{i+1})+\{\varepsilon_{i+1}(D_{i+2}-D_{i+1})+D_{i+1}(\varepsilon_{i+1}-\varepsilon_{i+2})\}s\}}} \right\} M_g \right\} \\
&\quad + \frac{1}{1-q^{-1-s\varepsilon_{r+1}}} \left\{ \frac{q^{-1-\varepsilon_{r+1}s-(a+b)(\varepsilon_r-\varepsilon_{r+1})-(D_{r+1}-D_r)-\{\varepsilon_{r+1}(D_{r+1}-D_r)-D_{r+1}(\varepsilon_r-\varepsilon_{r+1})\}s}}{1-q^{-\{(a+b)(\varepsilon_r-\varepsilon_{r+1})+(D_{r+1}-D_r)+\{\varepsilon_{r+1}(D_{r+1}-D_r)+D_{r+1}(\varepsilon_r-\varepsilon_{r+1})\}s\}}} \right. \\
&\quad \left. + \sum_{l=1}^{\varepsilon_r-\varepsilon_{r+1}-1} \frac{q^{-1-\varepsilon_{r+1}s-(a+b)l+[l\tau_{r+1}]-\{D_{r+1}l-\varepsilon_{r+1}[l\tau_{r+1}]\}s}}{1-q^{-\{(a+b)(\varepsilon_r-\varepsilon_{r+1})+(D_{r+1}-D_r)+\{\varepsilon_{r+1}(D_{r+1}-D_i)+D_{r+1}(\varepsilon_r-\varepsilon_{r+1})\}s\}}} \right\} M_{g_r} \\
&\quad + \sum_{i=1}^{r}\left\{ \frac{q^{-(a+b)(\varepsilon_i-\varepsilon_{i+1})-(D_{i+1}-D_i)-\{\varepsilon_i(D_{i+1}-D_i)-D_i(\varepsilon_i-\varepsilon_{i+1})\}s}}{1-q^{-\{(a+b)(\varepsilon_i-\varepsilon_{i+1})+(D_{i+1}-D_i)+\{\varepsilon_i(D_{i+1}-D_i)+D_i(\varepsilon_i-\varepsilon_{i+1})\}s\}}} \right. \\
&\quad \left. + \sum_{l=1}^{\varepsilon_i-\varepsilon_{i+1}-1} \frac{q^{-(a+b)l-[l\tau_i]-\{D_i l-\varepsilon_i[l\tau_i]\}s}}{1-q^{-\{(a+b)(\varepsilon_i-\varepsilon_{i+1})+(D_{i+1}-D_i)+\{\varepsilon_i(D_{i+1}-D_i)+D_i(\varepsilon_i-\varepsilon_{i+1})\}s\}}} \right\} M_G
\end{aligned}
$$

Here we introduce the following notation to obtain a compact form for the sum

$$B_{i,l} := (a+b)l + [l\tau_i] + 1 + s(D_{i+1}l + \varepsilon_{i+1}[l\tau_i] + \varepsilon_{i+1})$$

$$\rho_i := (a+b)(\varepsilon_i - \varepsilon_{i+1}) + (D_{i+1} - D_i)$$

$$\delta_i := D_{i+1}(\varepsilon_i - \varepsilon_{i+1}) + (D_{i+1} - D_i)\varepsilon_{i+1},$$

$$G_{i,l} := (a+b)l + [l\tau_{i+1}] + 1 + s(D_{i+1}l + \varepsilon_{i+1}[l\tau_{i+1}] + \varepsilon_{i+1})$$

$$\delta_i' := D_{i+1}(\varepsilon_{i+1} - \varepsilon_{i+2}) + (D_{i+2} - D_{i+1})\varepsilon_{i+1}.$$

Therefore

$$\sum_{m=1}^{\infty} q^{-(a+b)m-d_0ms} J_\theta(s,m,\chi_{triv})$$

$$= \sum_{i=0}^{r-1} M_g \left\{ \frac{q^{-1-\rho_i-\{\varepsilon_{i+1}-\delta_i\}s}}{(1-q^{-\rho_i-\delta_i s})(1-q^{-1-s\varepsilon_{i+1}})} + \sum_{l=1}^{\varepsilon_i-\varepsilon_{i+1}-1} \frac{q^{-B_{i,l}}}{(1-q^{-\rho_i-\delta_i s})(1-q^{-1-s\varepsilon_{i+1}})} \right\}$$

$$- \sum_{i=0}^{r-1} M_g \left\{ \frac{q^{-1-\rho_{i+1}-\{\varepsilon_{i+1}-\delta_i'\}s}}{(1-q^{-\rho_{i+1}-\delta_i' s})(1-q^{-1-s\varepsilon_{i+1}})} + \sum_{l=1}^{\varepsilon_{i+1}-\varepsilon_{i+2}-1} \frac{q^{-G_{i,l}}}{(1-q^{-\rho_{i+1}-\delta_i' s})(1-q^{-1-s\varepsilon_{i+1}})} \right\}$$

$$+ M_{g_r} \left\{ \frac{q^{-1-\rho_r-\{\varepsilon_{r+1}-\delta_r\}s}}{(1-q^{-\rho_r-\delta_r s})(1-q^{-1-s\varepsilon_{r+1}})} + \sum_{l=0}^{\varepsilon_r-\varepsilon_{r+1}-1} \frac{q^{-G_{r,l}}}{(1-q^{-\rho_r-\delta_r s})(1-q^{-1-s\varepsilon_{r+1}})} \right\}$$

$$+ \sum_{i=1}^{r} M_G \left\{ \frac{q^{-\rho_i-\delta_{i-1}' s}}{1-q^{-\rho_i-\delta_{i-1}' s}} + \sum_{l=0}^{\varepsilon_i-\varepsilon_{i+1}-1} \frac{q^{-G_{i-1,l}+(1+\varepsilon_{i+1}s)}}{1-q^{-\rho_i-\delta_{i-1}' s}} \right\}.$$

Similar equations holds in the case $\chi \neq \chi_{triv}$. It follows that real parts of the poles of

$$\sum_{\{\theta \in O_K^\times \mid f_0(1,\theta^a)=0\}} \left(\sum_{m=1}^{\infty} q^{-(a+b)m-d_0ms} J_\theta(s,m,\chi) \right),$$

belong to the set

$$\{-1\} \cup \left\{ -\frac{a+b}{d_0} \right\} \cup \bigcup_{\{\theta \in O_K^\times \mid f_0(1,\theta^a)=0\}} \mathcal{P}(\Gamma_{f,\theta}).$$

□

2.4.3 Examples

1. $f(x,y) = (y^3 - x^2)^2 + x^4y^4$. The polynomial $f(x,y) \in K[x,y]$ is a semiquasihomogeneous polynomial with respect to the weight $(3,2)$, which is the generator of the cone Δ_5, see example 2.3.1 and Table 1.1. We note that $f(x,y) = f_0(x,y) + f_1(x,y)$, where $f_0(x,y) = (y^3-x^2)^2$ and $f_1(x,y) = x^4y^4$, c.f. (2.1.1). In this case $\theta = 1$ is the only root of $f_0(1,y^3)$, thus $\Gamma^A(f) = \Gamma_{f,1}$.

   Since $f_0(t^3x, t^2y) = t^{12} f_0(x,y)$ and $f_1(t^3x,t^2y) = t^{20} f_1(x,y)$, the numerical data for $\Gamma_{f,1}$ are: $a = 3, b = 2, \mathcal{D}_1 = d_0 = 12, \tau_1 = 4, \varepsilon_1 = 2$, and $\mathcal{D}_2 = 20$, then the boundary of the arithmetic Newton polygon $\Gamma_{f,1}$ is formed by the straight segments

$$w_{0,1}(z) = 2z \ (0 \leqslant z \leqslant 4), \quad \text{and,} \quad w_{1,1}(z) = 8 \ (z \geqslant 4),$$

together with the half–line $\{(z, w) \in \mathbb{R}^2_+ | w = 0\}$. The face functions are

$$f_{(0,0)}(x, y) = (y^3 - x^2)^2, \quad f_{(4,8)}(x, y) = (y^3 - x^2)^2 + x^4y^4,$$

see figure 2.1: $\Gamma^A(f)$. Since that $f_{(4,8)}(x, y)$ does not have singular points on $K^{\times 2}$, $f(x, y)$ is arithmetically non-degenerate.

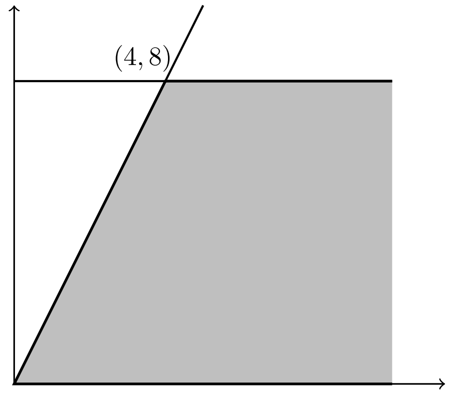

Figure 2.1: $\Gamma^A(f)$

According to Theorem 2.4.1, the real parts of the poles of $Z(s, f, \chi, \Delta_5)$ belong to the set $\{-1, -\frac{5}{12}, -\frac{1}{2}, -\frac{9}{20}\}$ cf. (2.3.4)–(2.3.9).

2. $g(x, y) = (y^3 - x^2)^2(y^3 - cx^2) + x^4y^4$. Let $c \in O_K^\times$ and $c \not\equiv 1 \mod \mathfrak{p}$ as in Example 2.3.2. The polynomial $g(x, y) \in K[x, y]$ is a semiquasihomogeneous polynomial with respect to the weight $(3, 2)$, which is the generator of the cone Δ_5, see Table 1.1. We note that $g(x, y) = g_0(x, y) + g_1(x, y)$, where $g_0(x, y) = (y^3 - x^2)^2(y^3 - cx^2)$ and $g_1(x, y) = x^4y^4$, c.f. (2.1.1). In this case $\theta = 1$ and $\theta = c$, are the roots of $g_0(1, y^3)$, thus $\Gamma^A(g) = \{\Gamma_{g,1}, \Gamma_{g,c}\}$.

Since $g_0(t^3x, t^2y) = t^{18}g_0(x, y)$ and $g_1(t^3x, t^2y) = t^{20}g_1(x, y)$, the numerical data for $\Gamma_{g,1}$ are: $a = 3, b = 2, \mathcal{D}_1 = d_0 = 18, \tau_1 = 1, \varepsilon_1 = 2$, and $\mathcal{D}_2 = 20$, then the boundary of the arithmetic Newton polygon $\Gamma_{g,1}$ is formed by the straight segments

$$w_{0,1}(z) = 2z \ (0 \leqslant z \leqslant 1), \quad \text{and,} \quad w_{1,1}(z) = 2 \ (z \geqslant 1),$$

together with the half–line $\{(z, w) \in \mathbb{R}_+^2 | w = 0\}$. The face functions are

$$g_{(0,0)}(x, y) = (y^3 - x^2)^2(y^3 - cx^2), \quad g_{(1,2)}(x, y) = (y^3 - x^2)^2(y^3 - cx^2) + x^4y^4,$$

see figure 2.2: $\Gamma_{g,1}$. Since $g_{(1,2)}(x, y)$ does not have singular points on $K^{\times 2}$, $g(x, y)$ is arithmetically non-degenerate with respect to $\Gamma_{g,1}$.

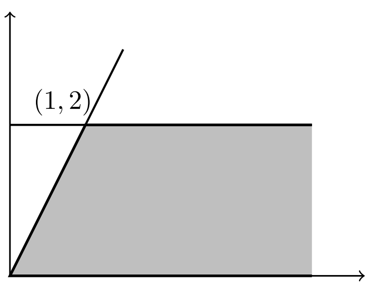

Figure 2.2: $\Gamma_{g,1}$

On the other hand, the numerical data for $\Gamma_{g,c}$ are: $a = 3, b = 2, \mathcal{D}_1 = d_0 = 18, \tau_1 = 2, \varepsilon_1 = 1$, and $\mathcal{D}_2 = 20$, then the boundary of the arithmetic Newton polygon $\Gamma_{g,c}$ is formed by the straight segments

$$w_{0,c}(z) = z \ (0 \leqslant z \leqslant 2), \quad \text{and,} \quad w_{1,c}(z) = 2 \ (z \geqslant 2),$$

together with the half–line $\{(z, w) \in \mathbb{R}_+^2 | w = 0\}$

The face functions are $g_{(0,0)}(x, y) = (y^3 - x^2)^2(y^3 - cx^2)$, $g_{(2,2)}(x, y) = (y^3 - x^2)^2(y^3 - cx^2) + x^4y^4$, see figure 2.3: $\Gamma_{g,c}$. Since $g_{(2,2)}(x, y)$ does not have singular points on $K^{\times 2}$, $g(x,y)$ is arithmetically non-degenerate with respect to $\Gamma_{g,c}$.

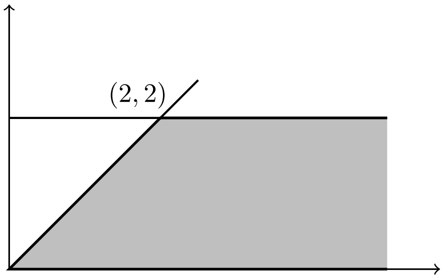

Figure 2.3: $\Gamma_{g,c}$

According to Theorem 2.4.1, the real parts of the poles of $Z(s,g,\chi,\Delta_5)$ belong to the set $\{-1, -\frac{5}{18}, -\frac{1}{2}, -\frac{6}{20}, -\frac{7}{20}\}$ cf. (2.3.10)-(2.3.14).

2.5 Local zeta functions for arithmetically non-degenerate polynomials

Take $f(x,y) \in K[x,y]$ be a non-constant polynomial satisfying $f(0,0) = 0$. Assume that

$$\mathbb{R}_+^2 = \{(0,0)\} \cup \bigcup_{\gamma \subset \Gamma^{geom}(f)} \Delta_\gamma, \qquad (2.5.1)$$

is a simplicial conical subdivision subordinated to $\Gamma^{geom}(f)$.

Let $a_\gamma = (a_1(\gamma), a_2(\gamma))$ be the perpendicular primitive vector to the edge γ of $\Gamma^{geom}(f)$, we also denote by $\langle a_\gamma, x\rangle = d_a(\gamma)$ the equation of the corresponding supporting line (cf. Section 1.3). We set

$$\mathcal{P}(\Gamma^{geom}(f)) := \left\{ -\frac{a_1(\gamma)+a_2(\gamma)}{d_a(\gamma)} \,\middle|\, \gamma \text{ is an edge of } \Gamma^{geom}(f), d_a(\gamma) \neq 0 \right\}.$$

Theorem 2.5.1. *Let $f(x,y) \in K[x,y]$ be a non-constant polynomial. If $f(x,y)$ is arithmetically modulo $\mathfrak{p}$ non-degenerate with respect to its arithmetic Newton polygon*

$\Gamma^{A}(f)$, *then the real parts of the poles of* $Z(s,f,\chi)$ *belong to the set*

$$\{-1\} \cup \mathcal{P}(\Gamma^{geom}(f)) \cup \mathcal{P}(\Gamma^{A}(f)).$$

In addition $Z(s,f,\chi)$ *vanishes for almost all* χ.

Proof. Consider the conical decomposition (2.5.1), then by (1.3.1) the problem of describe the poles of $Z(s,f,\chi)$ is reduced to the problem of describe the poles of $Z(s,f,\chi,O_K^{\times 2})$ and $Z(s,f,\chi,\Delta_\gamma)$, where γ is a proper face of $\Gamma^{geom(f)}$. By Lemma 1.2.3, the real part of the poles of $Z(s,f,\chi,O_K^{\times 2})$ is -1.

For the integrals $Z(s,f,\chi,\Delta_\gamma)$, we have two cases depending of the non degeneracy of f with respect to Δ_γ. If Δ_γ is a one–dimensional cone generated by $a_\gamma = (a_1(\gamma), a_2(\gamma))$, and $f_\gamma(x,y)$ does not have singularities on $(K^\times)^2$, then the real parts of the poles of $Z(s,f,\chi,\Delta_\gamma)$ belong to the set

$$\{-1\} \cup \left\{-\frac{a_1(\gamma)+a_2(\gamma)}{d_\gamma}\right\} \subseteq \{-1\} \cup \mathcal{P}(\Gamma^{geom}(f)).$$

If Δ_γ is a two–dimensional cone, $f_\gamma(x,y)$ is a monomial, and then it does not have singularities on the torus $(K^\times)^2$, in consequence $Z(s,f,\chi,\Delta_\gamma)$ is an entire function as can be deduced from [37, Proposition 4.1]. If Δ_γ is a one-dimensional cone, and $f_\gamma(x,y)$ has not singularities on $(O_K^\times)^2$, then $f(x,y)$ is a semiquasihomogeneous arithmetically non-degenerate polynomial, and thus by Theorem 2.4.1, the real parts of the poles of $Z(s,f,\chi,\Delta_\gamma)$ belong to the set

$$\{-1\} \cup \left\{-\frac{a_1(\gamma)+a_2(\gamma)}{d_\gamma}\right\} \cup \mathcal{P}(\Gamma^{A}(f)) \subseteq \{-1\} \cup \mathcal{P}(\Gamma^{geom}(f)) \cup \mathcal{P}(\Gamma^{A}(f)).$$

From these observations the real parts of the poles of $Z(s,f,\chi)$ belong to the set

$$\{-1\} \cup \mathcal{P}(\Gamma^{geom}(f)) \cup \mathcal{P}(\Gamma^{A}(f)).$$

Now we prove that $Z(s,f,\chi)$ vanishes for almost all χ. From (2.5.1) and (1.3.1) it is enough to show that the integrals $Z(s, f, \chi, \Delta_\gamma) = 0$ for almost all χ, to do so, we consider two cases. If f is non–degenerate with respect to Δ_γ, $Z(s, f, \chi, \Delta_\gamma) = 0$ for almost all χ, as follows from the proof of Theorem 1.3.1. On the other hand, when f is degenerate with respect to Δ_γ and Δ_γ is a one dimensional cone generated by a_γ, then $f(x, y)$ is a semiquasihomogeneous polynomial with respect to the weight a_γ , thus by Theorem 2.4.1, $Z(s, f, \chi, \Delta_\gamma) = 0$ when $\chi|_{1+\mathfrak{p}O_K} \neq \chi_{triv}$. If Δ_γ is a two dimensional cone, then γ is a point. Indeed, it is the intersection point of two edges τ and μ of $\Gamma^{geom}(f)$, and satisfies the equations:

$$\langle a_\tau, \gamma\rangle = d_a(\tau) \text{ and } \langle a_\mu, \gamma\rangle = d_a(\mu).$$

It follows that $f(x, y)$ is a semiquasihomogeneous polynomial with respect to the weight given by the barycenter of the cone: $\frac{a_\tau + a_\mu}{2}$. The weighted degree is $\frac{d_a(\tau)+d_a(\mu)}{2}$. Finally, we may use again Theorem 2.4.1 to obtain the required conclusion. □

Chapter 3

Exponential Sums $\mod \mathfrak{p}^m$

In this chapter we give some estimations for the asymptotic behavior of exponential sums $\mod \mathfrak{p}^m$ attached to arithmetically non-degenerate polynomial, see Theorem 3.1.1.

3.1 Exponential Sums

Let K be a non–Archimedean local field of arbitrary characteristic with valuation v, and take $f(x,y) \in K[x,y]$. The *exponential sum* attached to f is

$$E(z,f) := \int_{O_K^2} \Psi(zf(x,y)) \, |dxdy|,$$

for $z = u\mathfrak{p}^{-m}$ where $u \in O_K^\times$ and $m \in \mathbb{Z}$.

Lemma 3.1.1. *$E(z,f)$ can be thought of as an exponential sum.*

$$E(z,f) = q^{-2m} \sum_{(a,b)\in(O_K/P_K^m)^2} \Psi(zf(a,b)),$$

for $z = u\mathfrak{p}^{-m}$ where $u \in O_K^\times$ and $m \in \mathbb{Z}$ and $f(x,y) \in K[x,y]$.

Proof. In fact if we decompose O_K^2 as

$$O_K^2 = \bigsqcup_{(\overline{a},\overline{b})\in(O_K/\mathfrak{p}^m O_K)^2} (a,b) + (\mathfrak{p}^m O_K)^2,$$

we obtain,

$$E(z,f) = \sum_{(a,b)\in(O_K/\mathfrak{p}^m O_K)^2} \int_{((a,b)+\mathfrak{p}^m O_K)^2} \Psi(u\mathfrak{p}^{-m} f(x,y))|dxdy|, \qquad (3.1.1)$$

$$= q^{-2m} \sum_{(a,b)\in(O_K/\mathfrak{p}^m O_K)^2} \int_{O_K^2} \Psi(u\mathfrak{p}^{-m} f(a+\mathfrak{p}^m x_1, b+\mathfrak{p}^m y_1)|dx_1 dy_1|,$$

where $(x_1, y_1) \in O_K^2$. Now, by using the Taylor series for f around (a,b):

$$f(a+\mathfrak{p}^m x_1, b+\mathfrak{p}^m y_1) =$$
$$f(a,b) + \mathfrak{p}^m \left(\frac{\partial f}{\partial x_1}(a,b)x_1 + \frac{\partial f}{\partial y_1}(a,b)y_1 \right) + \mathfrak{p}^{m+1}(\textit{higher order terms}),$$

we get,

$$E(z,f) = q^{-2m} \sum_{(a,b)\in(O_K/P_K^m)^2} \Psi(zf(a,b)). \qquad (3.1.2)$$

□

Denef found the following nice relation between $E(z,f)$ and $Z(s,f,\chi)$.

We denote by $\text{Coeff}_{t^k} Z(s,f,\chi)$ the coefficient c_k in the power series expansion of $Z(s,f,\chi)$ in the variable $t = q^{-s}$.

Proposition 3.1.1 ([12, Proposition 1.4.4]). *With the above notation*

$$E(u\mathfrak{p}^{-m}, f) = Z(0, f, \chi_{triv}) + Coeff_{t^{m-1}} \frac{(t-q)Z(s, f, \chi_{triv})}{(q-1)(1-t)}$$
$$+ \sum_{\chi \neq \chi_{triv}} g_{\chi^{-1}} \chi(u) Coeff_{t^{m-c(\chi)}} Z(s, f, \chi),$$

where $c(\chi)$ denotes the conductor of χ and g_χ is the Gaussian sum

$$g_\chi = (q-1)^{-1} q^{1-c(\chi)} \sum_{x \in (O_K/P_K^{c(\chi)})^\times} \chi(x)\ \Psi(x/\mathfrak{p}^{c(\chi)}).$$

We recall here that the *critical set* of f is defined as

$$C_f := C_f(K) = \{(x, y) \in K^2 \mid \nabla f(x, y) = 0\}.$$

We also define

$$\beta_{\Gamma^{geom}} = \max_{\gamma \text{ edges of } \Gamma^{geom}(f)} \left\{ -\frac{a_1(\gamma) + a_2(\gamma)}{d_a(\gamma)} \middle| d_a(\gamma) \neq 0 \right\},$$

and

$$\beta_{\Gamma_\theta^A} := \max_{\theta \in R(f_0)} \{\mathcal{P} \mid \mathcal{P} \in \mathcal{P}(\Gamma_{f,\theta})\}.$$

Theorem 3.1.1. *Let $f(x, y) \in K[x, y]$ be a non constant polynomial which is arithmetically modulo $\mathfrak{p}$ non–degenerate with respect to its arithmetic Newton polygon. Assume that $C_f \subset f^{-1}(0)$ and assume all the notation introduced previously. Then the following assertions hold.*

1. *For $|z|$ big enough, $E(z, f)$ is a finite linear combination of functions of the form*

$$\chi(ac\ z)|z|^\lambda (\log_q\ |z|)^{j_\lambda},$$

with coefficients independent of z, and $\lambda \in \mathbb{C}$ a pole of $Z(s, f, \chi)$ (with $\chi|_{1+\mathfrak{p}O_K} =$

χ_{triv}) or $(1-q^{-s-1})Z(s,f,\chi_{triv})$, where

$$j_\lambda = \begin{cases} 0 & \text{if } \lambda \text{ is a simple pole} \\ 0,1 & \text{if } \lambda \text{ is a double pole.} \end{cases}$$

Moreover all the poles λ appear effectively in this linear combination.

2. *Assume that $\beta := \max\{\beta_{\Gamma^{geom}}, \beta_{\Gamma^A_\theta}\} > -1$. Then for $|z| > 1$, there exist a positive constant $C(K)$, such that*

$$|E(z,f)| \leqslant C(K)|z|^\beta \log_q \ |z|.$$

Proof. 1. The proof follows by writing $Z(s,f,\chi)$ in partial fractions and using Proposition 3.1.1 and Theorem 2.5.1. For $t = q^{-s}$,

$$\begin{aligned} Z(s,f,\chi) &= \sum_{m\geqslant 0} \int_{v(f(x))=m} \chi(ac\ f(x))\ |f(x)|^s\ dx, \\ &= \sum_{m\geqslant 0} Coeff_{t^m}(Z(s,f,\chi_{triv})) \cdot t^m. \end{aligned}$$

Note that $(1-q^{-s-1})Z(s,f,\chi_{triv})$ or $Z(s,f,\chi)$ may have simple poles or double poles. By Theorem 2.4.1, we know that the real part of the candidate poles λ of $Z(s,f,\chi)$ can be $\frac{\rho_i}{\delta_i}$, $\frac{\rho_{i+1}}{\delta_i'}$ or $\frac{1}{\varepsilon_i}$, where $\frac{\rho_i}{\delta_i} \neq \frac{\rho_{i+1}}{\delta_i'}$. Then by expanding $Z(s,f,\chi_{triv})$ in partial fractions over the complex numbers, we consider the following cases.

Case (i): Simple poles. In this case by using the identity
$1-q^{-\rho_i}t^{\delta_i} = (1-q^{\frac{-\rho_i}{\delta_i}}t) \prod_{\substack{\xi^{\delta_i}=1 \\ \xi\neq 1}} (1-\xi q^{\frac{-\rho_i}{\delta_i}}t)$, where $\xi \in \mathbb{C}$. Then we have

$$\frac{1}{1-q^{-\rho_i}t^{\delta_i}} = \sum_{\xi^{\delta_i}=1} c_\xi \sum_{l=0}^{\infty} q^{-\frac{\rho_i}{\delta_i}l} \xi^l t^l,$$

for some constant $c_\xi \in \mathbb{C}$.

Case (ii):Double poles. Here we have essentially two subcases. In the first case, when $\frac{1}{\varepsilon_i} \neq \frac{\rho_i}{\delta_i}$, we obtain

$$\frac{1}{(1-q^{-\rho_i}t^{\delta_i})(1-q^{-1}t^{\varepsilon_i})} = \sum_{\xi^{\delta_i}=1} c_\xi \left(\sum_{l=0}^{\infty} q^{-\frac{\rho_i}{\delta_i}l}\xi^l t^l\right) + \sum_{\xi^{\varepsilon_i}=1} e_\xi \left(\sum_{l=0}^{\infty} q^{-\frac{1}{\varepsilon_i}l}\xi^l t^l\right),$$

where c_ξ, e_ξ are constants.

The second case, is when $\frac{1}{\varepsilon_i} = \frac{\rho_i}{\delta_i}$. Here we have

$$\frac{1}{(1-q^{-\rho_i}t^{\delta_i})(1-q^{-1}t^{\varepsilon_i})} = \sum_{\substack{\xi^{\delta_i}=1 \\ \xi^{\varepsilon_i}=1}} \left(\frac{f_\xi}{\left(1-q^{\frac{-\rho_i}{\delta_i}}\xi t\right)^2} + \frac{h_\xi}{1-q^{\frac{-\rho_i}{\delta_i}}\xi t}\right)$$
$$+ \sum_{\substack{\xi^{\delta_i}=1 \\ \xi^{\varepsilon_i}\neq 1}} j_\xi \left(\sum_{l=0}^{\infty} q^{\frac{-\rho_i}{\delta_i}l}\xi^l t^l\right) + \sum_{\substack{\xi^{\varepsilon_i}=1 \\ \xi^{\delta_i}\neq 1}} k_\xi \left(\sum_{l=0}^{\infty} q^{\frac{-1}{\varepsilon_i}l}\xi^l t^l\right),$$

for some constants $f_\xi, h_\xi, j_\xi, k_\xi \in \mathbb{C}$. Note that

$$\frac{1}{\left(1-q^{\frac{-\rho_i}{\delta_i}}\xi t\right)^2} = \sum_{l=0}^{\infty}(l+1)q^{\frac{-\rho_i}{\delta_i}l}\xi^l t^l.$$

Therefore

$$Coeff_{t^m} Z(s,f,\chi_{triv}) = \sum_{\xi^{\delta_i}=1} \left(f_\xi(m+1)+h_\xi\right)\xi^m q^{-\frac{\rho_i}{\delta_i}m}.$$

We also note that for m big enough $Z(s,f,\chi)$ is rational function identically zero for almost all χ (Theorem 2.4.1), the series

$$\sum_{\chi\neq\chi_{triv}} g_{\chi^{-1}}\chi(u)\, Coeff_{t^{m-1}} Z(s,f,\chi)$$

is a finite sum. Then, $E(z, f)$ is asymptotically equal to

$$\sum_{\lambda} c_m \chi(ac\ z)|z|^{-\lambda}(\log_q |z|)^{j_\lambda},$$

where λ runs through all of the poles of $Z(s, f, \chi_{triv})$, and c_m are complex constant.

2. For $|z|$ big enough and $\beta > -1$, we have the estimation

$$|z|^{\lambda}(\log_q |z|)^{j_\lambda} \leqslant C(K)|z|^{\beta}(\log_q |z|),$$

which implies the desired estimation.

□

Appendix A

The local zeta function of $(y^3 - x^2)^2 + x^4y^4$

In this section we shall compute explicitly the local zeta functions for $f(x, y) = (y^3 - x^2)^2 + x^4y^4 \in K[x, y]$. We assume that the characteristic of the residue field of K is different from 2 and 3. This polynomial is degenerate with respect to its geometric Newton polygon in the sense of Kouchnirenko. We present the example 2.3.1 computed in full detail and we obtain an explicit list of candidates for the poles in terms of geometric data obtained from a family of arithmetic Newton polygons attached to the polynomial $f(x, y)$.

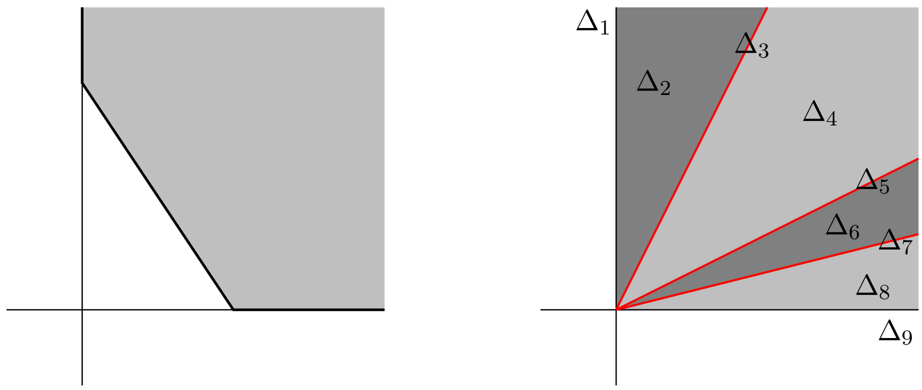

Figure A.1: $\Gamma^{geom}((y^3 - x^2)^2 + x^4y^4)$ and the conical partition of $\mathbb{R}^2_+$ induced by it.

The conical subdivision of $\mathbb{R}_+^2$ subordinated to the geometric Newton polygon of $f(x, y)$ is $\mathbb{R}_+^2 = \left\{(0,0) \cup \bigcup_{j=1}^{9} \Delta_j\right\}$.

Table A.1: Rational Simple Cones

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Cone	Generators
Δ_1	$(0,1)\mathbb{R}_+$
Δ_2	$(0,1)\mathbb{R}_+ + (1,1)\mathbb{R}_+$
Δ_3	$(1,1)\mathbb{R}_+$
Δ_4	$(1,1)\mathbb{R}_+ + (3,2)\mathbb{R}_+$
Δ_5	$(3,2)\mathbb{R}_+$
Δ_6	$(3,2)\mathbb{R}_+ + (2,1)\mathbb{R}_+$
Δ_7	$(2,1)\mathbb{R}_+$
Δ_8	$(2,1)\mathbb{R}_+ + (1,0)\mathbb{R}_+$
Δ_9	$(1,0)\mathbb{R}_+$

A.1 Computation of $Z(s, f, \chi, \Delta_i)$

These integrals correspond to the case in which f is non-degenerate in the sense of Kouchnirenko on Δ_i, for $i = 1, 2, 3, 4, 6, 7, 8, 9$, as in section 1.3. The integrals can be calculated as follows.

1. Case $Z(s, f, \chi, \Delta_1)$.

$$Z(s, f, \chi, \Delta_1) = \sum_{n=1}^{\infty} \int_{O_K^\times \times \mathfrak{p}^n O_K^\times} \chi(ac\, f(x, y))|f(x, y)|^s |dxdy|.$$

$$Z(s, f, \chi, \Delta_1) =$$
$$= \sum_{n=1}^{\infty} \int_{O_K^{\times} \times \mathfrak{p}^n O_K^{\times}} \chi(ac\ (y^3 - x^2)^2 + x^4y^4)|(y^3 - x^2)^2 + x^4y^4|^s |dxdy|,$$
$$= \sum_{n=1}^{\infty} q^{-n} \int_{O_K^{\times 2}} \chi(ac\ (\mathfrak{p}^{3n}y^3 - x^2)^2 + \mathfrak{p}^{4n}x^4y^4)|(\mathfrak{p}^{3n}y^3 - x^2)^2 + \mathfrak{p}^{4n}x^4y^4|^s |dxdy|,$$
$$= \sum_{n=1}^{\infty} q^{-n} \int_{O_K^{\times 2}} \chi(ac\ (g_1(x, y)))|(g_1(x, y))|^s |dxdy|,$$

where $g_1(x, y) = (\mathfrak{p}^{3n}y^3 - x^2)^2 + \mathfrak{p}^{4n}x^4y^4$, with $\overline{g}_1(x, y) = x^4$.

Note that we can write $O_K^{\times 2}$ as follows

$$O_K^{\times 2} = \dot{\bigcup_{(\overline{a}, \overline{b}) \in (\mathbb{F}_q^{\times})^2}} (a, b) + (\mathfrak{p}O_K)^2. \tag{A.1.1}$$

Thus we can write

$$Z(s, f, \chi, \Delta_1) = \sum_{n=1}^{\infty} q^{-n} \sum_{(\overline{a}, \overline{b}) \in (\mathbb{F}_q^{\times})^2} \int_{(a,b)+(\mathfrak{p}O_K)^2} \chi(ac(g_1(x, y)))|g_1(x, y)|^s |dxdy|,$$
$$= \sum_{n=1}^{\infty} q^{-n-2} \sum_{(\overline{a}, \overline{b}) \in (\mathbb{F}_q^{\times})^2} \int_{O_K^2} \chi(ac(g_1(a + \mathfrak{p}x, b + \mathfrak{p}y)))|g_1(a + \mathfrak{p}x, b + \mathfrak{p}y)|^s |dxdy|.$$

Set $\mathbf{x} = (x_1, x_2)$and $\mathbf{c} = (c_1, c_2)$. The Taylor series expansion of $g(\mathbf{c} + \mathfrak{p}\mathbf{x})$ around the origin is,

$$g(\mathbf{c} + \mathfrak{p}\mathbf{x}) = g(\mathbf{c}) + \mathfrak{p} \left(\frac{\partial g}{\partial x_1} \mathbf{c} x_1 + \frac{\partial g}{\partial x_2} \mathbf{c} x_2 \right) + \mathfrak{p}^2 (\textit{higher order terms}) \tag{A.1.2}$$

By using equation (A.1.2) an the fact that $\frac{\partial g_1}{\partial x}(\overline{a},\overline{b}) = 4\overline{a}^3 \neq 0$, we can change variables in the previous integral as follows

$$\begin{cases} z_1 = & \frac{g_1(a+\mathfrak{p}x,b+\mathfrak{p}y)-g_1(a,b)}{\mathfrak{p}}, \\ z_2 = & y, \end{cases} \tag{A.1.3}$$

$z=(z_1,z_2)$ is an special restricted power series (SRP) in (x,y). (c.f [22], Lemma 7.4.3).

We use the change of variables above and we obtain that, the mapping $(x,y)\to(z_1,z_2)$ on O_K^2 into O_K^2 preserves the Haar measure.

$$Z(s,f,\chi,\Delta_1) =$$

$$\sum_{n=1}^{\infty} q^{-n-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K^2} \chi(ac(g_1(a+\mathfrak{p}x,b+\mathfrak{p}y))|g_1(a+\mathfrak{p}x,b+\mathfrak{p}y)|^s|dxdy|,$$

$$= \sum_{n=1}^{\infty} q^{-n-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K} \chi(ac\ (g_1(a,b)+\mathfrak{p}z_1))|g_1(a,b)+\mathfrak{p}z_1|^s|dz_1|,$$

$$= \sum_{n=1}^{\infty} q^{-n-2} I_{\Delta_1}(s,(a,b)),$$

where, $I_{\Delta_1}(s,(a,b)) = \sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}} \int_{O_K} \chi(ac\ (g_1(a,b)+\mathfrak{p}z_1))|g_1(a,b)+\mathfrak{p}z_1|_K^s|dz_1|$.

For to compute $I_{\Delta_1}(s,(a,b))$ we find that $N = Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2 : \overline{a}^4=0\}=0$, then we use the Lemma 1.2.2 and we have that

$$I_{\Delta_1}(s,(a,b)) = \begin{cases} (q-1)^2 & if\ \ \chi=\chi_{triv} \\ \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \overline{\chi}(\overline{a}^4) & if\ \ \overline{\chi}=\chi_{triv} \\ 0 & all\ other\ cases, \end{cases}$$

where $\overline{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}_q$.

Now since that,

$$\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \overline{\chi}(\overline{a}^4) = \begin{cases} (q-1)^2 & \textit{if } \overline{\chi}^4 = \chi_{triv} \\ (q-1)\cdot 0 = 0 & \textit{if } \overline{\chi}^4 \neq \chi_{triv}, \end{cases}. \tag{A.1.4}$$

we obtain,

$$I_{\Delta_1}(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if } \chi = \chi_{triv} \\ (q-1)^2 & \textit{if } \overline{\chi}^4 = \chi_{triv} \\ 0 & \textit{all other cases.} \end{cases}$$

Since that $\overline{\chi}^4 = \chi_{triv}$ and $\chi|_U = \chi_{triv}$ is equivalent to $\chi^4 = \chi_{triv}$, we have that $Z(s,f,\chi,\Delta_1) = \sum_{n=1}^{\infty} q^{-n-2} I_{\Delta_1}(s,(a,b))$ so we get,

$$Z(s,f,\chi,\Delta_1) = \begin{cases} q^{-1}(1-q^{-1}) & \textit{if } \chi = \chi_{triv} \\ q^{-1}(1-q^{-1}) & \textit{if } \overline{\chi}^4 = \chi_{triv} \\ 0 & \textit{all other cases,} \end{cases}$$

where $U = 1 + \mathfrak{p}O_K$.

2. Case $Z(s,f,\chi,\Delta_2)$.

$$Z(s,f,\chi,\Delta_2) =$$
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_{\mathfrak{p}^m O_K^{\times}\times\mathfrak{p}^{n+m}O_K^{\times}}\chi(ac(f(x,y))|f(x,y)|^s|dxdy|,$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{-2m-n}\int_{O_K^{\times 2}}\chi(ac\ (f(\mathfrak{p}^m x,\mathfrak{p}^{n+m}y))|f(\mathfrak{p}^m x,\mathfrak{p}^{n+m}y)|^s|dxdy|,$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{-2m-n-4ms}\int_{O_K^{\times 2}}\chi(ac\ (g_2(x,y)))|g_2(x,y)|^s|dxdy|.$$

Since that polynomial $g_2(x,y)=(\mathfrak{p}^{3n+m}y^3-x^2)^2+\mathfrak{p}^{4n+4m}x^4y^4$, we have that $\overline{g}_2(x,y)=x^4$.

By using equation (A.1.1), so we obtain that,

$$Z(s,f,\chi,\Delta_2) =$$
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{-2m-n-4ms}\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2}\int_{(a,b)+(\mathfrak{p}O_K)^2}\chi(ac\ (g_2(x,y))|g_2(x,y)|^s|dxdy|,$$
$$=\sum_{m=n=1}^{\infty}q^{-2m-n-4ms-2}\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2}\int_{O_K^2}\chi(ac\ (g_2(a+\mathfrak{p}x,b+\mathfrak{p}y))|g_2(a+\mathfrak{p}x,b+\mathfrak{p}y)|^s|dxdy|.$$

Then we apply the change variables (A.1.3) to function g_2 and since that $\frac{\partial\overline{g_2}}{\partial x}(\overline{a},\overline{b})=4\overline{a}^3\neq 0$, we obtain,

$$Z(s, f, \chi, \Delta_2) =$$
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{-2m-n-4ms-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K} \chi(ac\ (g_2(a,b) + \mathfrak{p}z_1))|g_2(a,b) + \mathfrak{p}z_1)|^s|dz_1|,$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{-2m-n-4ms-2} I_{\Delta_2}(s,(a,b)),$$

where $I_{\Delta_2}(s,(a,b)) = \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K} \chi(ac\ (g_2(a,b) + \mathfrak{p}z_1))|g_2(a,b) + \mathfrak{p}z_1|^s|dz_1|$, and since that, $N = Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2 : \overline{a}^4 = 0\} = 0$, then by applying the same procedure above we obtain

$Z(s, f, \chi, \Delta_2) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{-2m-n-4ms-2} I_{\Delta_2}(s,(a,b))$ so we get,

$$Z(s, f, \chi, \Delta_2) = \begin{cases} \frac{q^{-3-4s}(1-q^{-1})}{(1-q^{-2-4s})} & \textit{if } \chi = \chi_{triv}, \\ \\ \frac{q^{-3-4s}(1-q^{-1})}{(1-q^{-2-4s})} & \textit{if } \chi^4 = \chi_{triv}, \chi|_U = \chi_{triv} \\ \\ 0 & \textit{all other cases}, \end{cases}$$

3. Case $Z(s, f, \chi, \Delta_3)$.

$$Z(s, f, \chi, \Delta_3)$$
$$= \sum_{n=1}^{\infty} \int_{\mathfrak{p}^n O_K^{\times} \times \mathfrak{p}^n O_K^{\times}} \chi(ac\ (f(x,y))|f(x,y)|^s|dxdy|,$$
$$= \sum_{n=1}^{\infty} q^{-2n} \int_{O_K^{\times 2}} \chi(ac\ (\mathfrak{p}^{3n}y^3 - \mathfrak{p}^{2n}x^2)^2 + \mathfrak{p}^{8n}x^4y^4)|(\mathfrak{p}^{3n}y^3 - \mathfrak{p}^{2n}x^2)^2 + \mathfrak{p}^{8n}x^4y^4|^s|dxdy|,$$
$$= \sum_{n=1}^{\infty} q^{-2n-4ns} \int_{O_K^{\times 2}} \chi(ac\ (\mathfrak{p}^{n}y^3 - x^2)^2 + \mathfrak{p}^{4n}x^4y^4)|(\mathfrak{p}^{n}y^3 - x^2)^2 + \mathfrak{p}^{4n}x^4y^4|^s|dxdy|,$$
$$= \sum_{n=1}^{\infty} q^{-2n-4ns} \int_{O_K^{\times 2}} \chi(ac(g_3(x,y)))|g_3(x,y)|^s|dxdy|,$$

where $g_3(x,y)=(\mathfrak{p}^n y^3-x^2)^2+\mathfrak{p}^{4n}x^4y^4$, we have $\overline{g}_3(x,y)=x^4$, then the origin of K is the only singular point of $g_3(x,y)$ over $(\mathbb{F}_q^{\times})^2$.

By using equation (A.1.1), so we obtain that,

$$Z(s,f,\chi,\Delta_3)=$$

$$\sum_{n=1}^{\infty} q^{-2n-4ns} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int\limits_{(a,b)+(\mathfrak{p}O_K)^2} \chi(ac\ g_3(x,y))|g_3(x,y)|^s|dxdy|,$$

$$=\sum_{n=1}^{\infty} q^{-2n-4ns-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int\limits_{O_K^2} \chi(ac\ g_3(a+\mathfrak{p}x,b+\mathfrak{p}y))|g_3(a+\mathfrak{p}x,b+\mathfrak{p}y)|^s|dxdy|.$$

Then we apply the change variables (A.1.3) to function g_3 and since that $\frac{\partial \overline{g_3}}{\partial x}(\overline{a},\overline{b})=4\overline{a}^3\neq 0$, we obtain,

$$Z(s,f,\chi,\Delta_3)=$$

$$\sum_{n=1}^{\infty} q^{-2n-4ns-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int\limits_{O_K^2} \chi(ac\ (g_3(a+\mathfrak{p}x,b+\mathfrak{p}y))|g_3(a+\mathfrak{p}x,b+\mathfrak{p}y)|^s|dxdy|,$$

$$=\sum_{n=1}^{\infty} q^{-2n-4ns-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int\limits_{O_K} \chi(ac\ (g_3(a,b)+\mathfrak{p}z_1))|g_3(a,b)+\mathfrak{p}z_1)|^s|dz_1|,$$

$$=\sum_{n=1}^{\infty} q^{-2n-4ns-2} I_{\Delta_3}(s,(a,b)),$$

where $I_{\Delta_3}(s,(a,b))=\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int\limits_{O_K} \chi(ac\ (g_3(a,b)+\mathfrak{p}z_1))|g_3(a,b)+\mathfrak{p}z_1)|^s|dz_1|$.

Then since that $N=Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2:\overline{a}^4=0\}=0$, and by (A.1.3) we obtain,

$$Z(s,f,\chi,\Delta_3) = \begin{cases} \frac{q^{-2-4s}(1-q^{-1})^2}{(1-q^{-2-4s})} & \textit{if } \chi = \chi_{triv}, \\ \frac{q^{-2-4s}(1-q^{-1})^2}{(1-q^{-2-4s})} & \textit{if } \chi^4 = \chi_{triv}, \chi|_U = \chi_{triv} \\ 0 & \textit{all other cases}, \end{cases}$$

where $U = 1 + \mathfrak{p}O_K$.

4. Case $Z(s,f,\chi,\Delta_4)$.

$$Z(s,f,\chi,\Delta_4) =$$
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \int\limits_{\mathfrak{p}^{n+3m}O_K^{\times}\times\mathfrak{p}^{n+2m}O_K^{\times}} \chi(ac\ f(x,y))|f(x,y)|^s|dxdy|,$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{-2n-5m}\int_{O_K^{\times 2}} \mathcal{X}((\mathfrak{p}^{3n+6m}y^3 - \mathfrak{p}^{2n+6m}x^2)^2 + \mathfrak{p}^{8n+20m}x^4y^4)|dxdy|,$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{(-2-4s)n+(-5-12s)m}\int\limits_{O_K^{\times 2}} \chi(ac(g_4(x,y)))|g_4(x,y)|^s|dxdy|.$$

where

$$\mathcal{X}((\mathfrak{p}^{3n+6m}y^3 - \mathfrak{p}^{2n+6m}x^2)^2 + \mathfrak{p}^{8n+20m}x^4y^4) =$$
$$\chi(ac\ ((\mathfrak{p}^{3n+6m}y^3 - \mathfrak{p}^{2n+6m}x^2)^2 + \mathfrak{p}^{8n+20m}x^4y^4))\times$$
$$|(\mathfrak{p}^{3n+6m}y^3 - \mathfrak{p}^{2n+6m}x^2)^2 + \mathfrak{p}^{8n+20m}x^4y^4)|,^s$$

and the polynomial $g_4(x,y) = (\mathfrak{p}^n y^3 - x^2)^2 + \mathfrak{p}^{4n+8m}x^4y^4$, then we have $\overline{g}_4(x,y) = x^4$, therefore the origin of K is the only singular point of $g_4(x,y)$ over $(\mathbb{F}_q^{\times})^2$.

We obtain that,

$$Z(s,f,\chi,\Delta_4)=$$
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{(-2-4s)n+(-5-12s)m}\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2}\int_{(a,b)+(\mathfrak{p}O_K)^2}\chi(ac\ g_4(x,y))|g_4(x,y)|^s|dxdy|.$$

Then since that $\frac{\partial\overline{g_4}}{\partial x}(\overline{a},\overline{b})=4\overline{a}^3\neq 0$, we obtain,

$$Z(s,f,\chi,\Delta_4)=$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{(-2-4s)n+(-5-12s)m-2}\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\int_{O_K}\chi(ac\ (g_4(a,b)+\mathfrak{p}z_1))|g_4(a,b)+\mathfrak{p}z_1|^s|dz_1|,$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{(-2-4s)n+(-5-12s)m-2}I_{\Delta_4}(s,(a,b)),$$

where $I_{\Delta_4}=\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\int_{O_K}\chi(ac\ (g_4(a,b)+\mathfrak{p}z_1))|g_4(a,b)+\mathfrak{p}z_1|^s|dz_1|$, then since that $N=Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2:\overline{a}^4=0\}=0$, and by applying (A.1.4) to $I_{\Delta_4}(s,(a,b))$, finally we obtain

$$Z(s,f,\chi,\Delta_4)=\begin{cases}\frac{q^{-7-16s}(1-q^{-1})^2}{(1-q^{-2-4s})(1-q^{-5-12s})} & \textit{if } \chi=\chi_{triv},\\ \\ \frac{q^{-7-16s}(1-q^{-1})^2}{(1-q^{-2-4s})(1-q^{-5-12s})} & \textit{if } \chi^4=\chi_{triv},\chi|_U=\chi_{triv}\\ \\ 0 & \textit{all other cases,}\end{cases}$$

5. Case $Z(s,f,\chi,\Delta_6)$.

$$Z(s,f,\chi,\Delta_6)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_{\mathfrak{p}^{3n+2m}O_K^{\times}\times\mathfrak{p}^{2n+m}O_K^{\times}}\chi(ac\ f(x,y))|f(x,y)|^s|dxdy|,$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{(-5-12s)n+(-3-6s)m}\int_{O_K^{\times 2}}\chi(ac(g_6(x,y)))|g_6(x,y)|^s\ |dxdy|,$$

where $g_6(x,y)=(y^3-\mathfrak{p}x^2)^2+\mathfrak{p}^{8n+6m}x^4y^4$ we have $\overline{g}_6(x,y)=y^6$ and we obtain that the origin of K is the only singular point of $g_6(x,y)$ over $(\mathbb{F}_q^{\times})^2$.

Now we obtain that,

$$Z(s,f,\chi,\Delta_6)=$$
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{(-5-12s)n+(-3-6s)m}\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2}\int_{(a,b)+(\mathfrak{p}O_K)^2}\chi(ac\ g_6(x,y))|g_6(x,y)|^s|dxdy|,$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{(-5-12s)n+(-3-6s)m-2}\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2}\int_{O_K^2}\mathcal{X}(ac(g_6(a+\mathfrak{p}x,b+\mathfrak{p}y)))|dxdy|.$$

where $\mathcal{X}(ac(g_6(a+\mathfrak{p}x,b+\mathfrak{p}y)))=\chi(ac(g_6(a+\mathfrak{p}x,b+\mathfrak{p}y)))|g_6(a+\mathfrak{p}x,b+\mathfrak{p}y)|^s$.

Then we apply the change variables (A.1.3) to function g_6 and since that $\frac{\partial \overline{g_6}}{\partial y}(\overline{a},\overline{b})=6\overline{b}^5\neq 0$, we obtain,

$$Z(s,f,\chi,\Delta_6)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{(-5-12s)n+(-3-6s)m-2}I_{\Delta_6}(s,(a,b)),$$

where $I_{\Delta_6}(s,(a,b))=\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2}\int_{O_K}\chi(ac\ (g_6(a,b)+\mathfrak{p}z_1))|g_6(a,b)+\mathfrak{p}z_1)|^s|dz_1|$, then since that $N=Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2:\overline{b}^6=0\}=0$, we get,

$$I_{\Delta_6}(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if} \quad \overline{\chi} = \chi_{triv} \\ \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \overline{\chi}(\overline{b}^6) & \textit{if} \quad \overline{\chi} = \chi_{triv} \\ & \\ 0 & \textit{all other cases,} \end{cases}$$

where $\overline{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}_q$, thus we resolving the sum

$$\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \overline{\chi}(\overline{b}^6) = \begin{cases} (q-1)\cdot 0 = 0 & \textit{if} \;\; \overline{\chi}^6 \neq \chi_{triv} \\ (q-1)^2 & \textit{if} \;\; \overline{\chi}^6 = \chi_{triv}, \end{cases} \tag{A.1.5}$$

and we have that,

$$I_{\Delta_6}(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if} \;\; \chi = \chi_{triv} \\ & \\ (q-1)^2 & \textit{if} \;\; \overline{\chi}^6 = \chi_{triv} \\ & \\ 0 & \textit{all other cases,} \end{cases}$$

Finally, since that $\overline{\chi}^6 = \chi_{triv}$ and $\chi|_U = \chi_{triv}$ is equivalent to $\chi^6 = \chi_{triv}$ where $U = 1 + \mathfrak{p}O_K$, we obtain

$$Z(s,f,\chi,\Delta_6 = \begin{cases} \frac{q^{-8-18s}(1-q^{-1})^2}{(1-q^{-3-6s})(1-q^{-5-12s})} & \textit{if}\, \chi = \chi_{triv}, \\ & \\ \frac{q^{-8-18s}(1-q^{-1})^2}{(1-q^{-3-6s})(1-q^{-5-12s})} & \textit{if}\, \chi^6 = \chi_{triv}, \chi|_U = \chi_{triv} \\ & \\ 0 & \textit{all other cases,} \end{cases}$$

6. Case $Z(s,f,\chi,\Delta_7)$.

$$Z(s,f,\chi,\Delta_7) = \sum_{n=1}^{\infty} \int\limits_{\mathfrak{p}^{2n}O_K^\times \times \mathfrak{p}^n O_K^\times} \chi(ac\ f(x,y))|f(x,y)|^s|dxdy|,$$

$$= \sum_{n=1}^{\infty} q^{-3n} \int\limits_{O_K^{\times 2}} \chi(ac\ (\mathfrak{p}^{3n}y^3 - \mathfrak{p}^{4n}x^2)^2 + \mathfrak{p}^{12n}x^4y^4)|(\mathfrak{p}^{3n}y^3 - \mathfrak{p}^{4n}x^2)^2 + \mathfrak{p}^{12n}x^4y^4|^s|dxdy|,$$

$$= \sum_{n=1}^{\infty} q^{-3n-6ns} \int\limits_{O_K^{\times 2}} \chi(ac\ (y^3 - \mathfrak{p}^{n}x^2)^2 + \mathfrak{p}^{6n}x^4y^4)|(y^3 - \mathfrak{p}^{n}x^2)^2 + \mathfrak{p}^{6n}x^4y^4|^s|dxdy|.$$

$$= \sum_{n=1}^{\infty} q^{-3n-6ns} \int_{O_K^{\times 2}} \chi(ac(g_7(x,y))|g_7(x,y)|^s\ \ |dxdy|.$$

Since that polynomial $g_7(x,y) = (y^3 - \mathfrak{p}^n x^2)^2 + \mathfrak{p}^{6n}x^4y^4$, we have $\overline{g}_7(x,y) = y^6$, then the origin of K is the only singular point of $g_7(x,y)$ over $(\mathbb{F}_q^\times)^2$.

We obtain that,

$$Z(s,f,\chi,\Delta_7) = \sum_{n=1}^{\infty} q^{-3n-6ns} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int\limits_{(a,b)+(\mathfrak{p}O_K)^2} \chi(ac\ g_7(x,y))|g_7(x,y)|^s|dxdy|$$

$$= \sum_{n=1}^{\infty} q^{-3n-6ns-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int\limits_{O_K^2} \chi(ac\ g_7(a+\mathfrak{p}x, b+\mathfrak{p}y))|g_7(a+\mathfrak{p}x, b+\mathfrak{p}y)|^s|dxdy|.$$

Since that $\frac{\partial \overline{g_7}}{\partial y}(\overline{a},\overline{b}) = 6\overline{b}^5 \neq 0$, we obtain,

$$Z(s,f,\chi,\Delta_7) = \sum_{n=1}^{\infty} q^{-3n-6ns-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int\limits_{O_K^2} \chi(ac\ (g_7(a,b)+\mathfrak{p}z_1))|g_7(a,b)+\mathfrak{p}z_1)|^s|dz_1|,$$

$$= \sum_{n=1}^{\infty} q^{-3n-6ns-2} I_{\Delta_7}(s,(a,b)),$$

where $I_{\Delta_7}(s,(a,b)) = \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int\limits_{O_K} \chi(ac\ (g_7(a,b)+\mathfrak{p}z_1))|g_7(a,b)+\mathfrak{p}z_1)|^s|dz_1|,$

then we applying the Lemma 1.2.2, and since that $N = Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2 : b^6 = 0\} = 0$,

then by applying (A.1.5) to $I_{\Delta_7}(s,(a,b))$ and we obtain

$$Z(s,f,\chi,\Delta_7) = \begin{cases} \frac{q^{-3-6s}(1-q^{-1})^2}{(1-q^{-3-6s})}, & if\ \chi = \chi_{triv}, \\ \frac{q^{-3-6s}(1-q^{-1})^2}{(1-q^{-3-6s})}, & if\ \chi^6 = \chi_{triv}, \chi|_U = \chi_{triv}, \\ 0, & all\ other\ cases. \end{cases}$$

7. Case $Z(s,f,\chi,\Delta_8)$.

$$Z(s,f,\chi,\Delta_8) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \int_{\mathfrak{p}^{2n+m}O_K^{\times}\times\mathfrak{p}^nO_K^{\times}} \chi(ac\ f(x,y))|f(x,y)|^s|dxdy|,$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{(-3-6s)n-m} \int_{O_K^{\times 2}} \chi(ac(g_8(x,y))|g_8(x,y)|^s|dxdy|.$$

Where $g_8(x,y) = (y^3 - \mathfrak{p}^{n+2m}x^2)^2 + \mathfrak{p}^{6n+4m}x^4y^4$ we have $\overline{g}_8(x,y) = y^6$, then the origin of K is the only singular point of $g_8(x,y)$, over $(\mathbb{F}_q^{\times})^2$.

By using equation (A.1.1), so we obtain that,

$$Z(s,f,\chi,\Delta_8) =$$
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{(-3-6s)n-m-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{(a,b)+(\mathfrak{p}O_K)^2} \chi(ac\ g_8(x,y))|g_8(x,y)|^s|dxdy| =$$
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{(-3-6s)n-m-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K^2} \chi(ac\ g_8(a+\mathfrak{p}x, b+\mathfrak{p}y))|g_8(a+\mathfrak{p}x, b+\mathfrak{p}y)|^s|dxdy|.$$

Then we apply the change variables (A.1.3) to function g_8 and since that $\frac{\partial \overline{g_8}}{\partial y}(\overline{a},\overline{b}) =$

$6\overline{b}^5 \neq 0$, consequently

$$Z(s,f,\chi,\Delta_8) =$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{(-3-6s)n-m-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K^2} \chi(ac\ g_8(a+\mathfrak{p}x, b+\mathfrak{p}y))|g_8(a+\mathfrak{p}x,b+\mathfrak{p}y)|^s|dxdy|,$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{(-3-6s)n-m-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K} \chi(ac\ (g_8(a,b)+\mathfrak{p}z_1))|g_8(a,b)+\mathfrak{p}z_1)|^s|dz_1|,$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{(-3-6s)n-m-2} I_{\Delta_8}(s,(a,b)),$$

where $I_{\Delta_8}(s,(a,b)) = \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K} \chi(ac\ (g_8(a,b)+\mathfrak{p}z_1))|g_8(a,b)+\mathfrak{p}z_1)|^s|dz_1|$, then $N = Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2 : \overline{b}^6 = 0\} = 0$, thus we applying the Lemma 1.2.2 and (A.1.5), it follows that

$$Z(s,f,\chi,\Delta_8 = \begin{cases} \frac{q^{-4-6s}(1-q^{-1})}{(1-q^{-3-6s})}, & \textit{if } \chi = \chi_{triv}, \\ \\ \frac{q^{-4-6s}(1-q^{-1})}{(1-q^{-3-6s})}, & \textit{if } \chi^6 = \chi_{triv}, \chi|_U = \chi_{triv} \\ \\ 0 & \textit{all other cases,} \end{cases}$$

note that $\overline{\chi}^6 = \chi_t riv$ and $\overline{\chi}|_U = \chi_{triv}$, $U = 1+\mathfrak{p}O_K$ is equivalent to $\chi^6 = \chi_{triv}$.

8. Case $Z(s,f,\chi,\Delta_9)$.

$$Z(s,f,\chi,\Delta_9) = \sum_{n=1}^{\infty} \int_{\mathfrak{p}^n O_K^{\times}\times O_K^{\times}} \chi(ac\ f(x,y))|f(x,y)|^s|dxdy|,$$

$$= \sum_{n=1}^{\infty} q^{-n} \int_{O_K^{\times 2}} \chi(ac\ (y^3-\mathfrak{p}^{2n}x^2)^2+\mathfrak{p}^{4n}x^4y^4)|(y^3-\mathfrak{p}^{2n}x^2)^2+\mathfrak{p}^{4n}x^4y^4|^s|dxdy|,$$

$$= \sum_{n=1}^{\infty} q^{-n} \int_{O_K^{\times 2}} \chi(ac(g_9(x,y)))|g_9(x,y)|^s\ |dxdy|.$$

Since that the polynomial $g_9(x, y) = (y^3 - \mathfrak{p}^{2n}x^2)^2 + \mathfrak{p}^{4n}x^4y^4$, we have $\overline{g}_9(x, y) = y^6$, thus the origin of K is the only singular point of $g_9(x, y)$ over $(\mathbb{F}_q^\times)^2$.

By using equation (A.1.1), $Z(s, f, \chi, \Delta_9)$ becomes

$$Z(s, f, \chi, \Delta_9) = \sum_{n=1}^{\infty} q^{-n} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int_{(a,b)+(\mathfrak{p}O_K)^2} \chi(ac\ g_9(x, y))|g_9(x, y)|^s|dxdy|,$$

$$= \sum_{n=1}^{\infty} q^{-n-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int_{O_K^2} \chi(ac\ g_9(a + \mathfrak{p}x, b + \mathfrak{p}y))|g_9(a + \mathfrak{p}x, b + \mathfrak{p}y)|^s|dxdy|.$$

Then we apply the change variables (A.1.3) to function g_9 and since that $\frac{\partial \overline{g_9}}{\partial y}(\overline{a}, \overline{b}) = 6\overline{b}^5 \neq 0$, we obtain,

$$Z(s, f, \chi, \Delta_9) =$$

$$\sum_{n=1}^{\infty} q^{-n-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int_{O_K^2} \chi(ac\ g_9(a + \mathfrak{p}x, b + \mathfrak{p}y))|g_9(a + \mathfrak{p}x, b + \mathfrak{p}y)|^s|dxdy|$$

$$= \sum_{n=1}^{\infty} q^{-n-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int_{O_K} \chi(ac\ (g_9(a, b) + \mathfrak{p}z_1))|g_9(a, b) + \mathfrak{p}z_1|^s|dz_1|$$

$$= \sum_{n=1}^{\infty} q^{-n-2} I_{\Delta_9}(s, (a, b)),$$

where $I_{\Delta_9}(s, (a, b)) = \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int\limits_{O_K} \chi(ac\ (g_9(a, b) + \mathfrak{p}z_1))|g_9(a, b) + \mathfrak{p}z_1|^s|dz_1|$, then given that $N = Card\{(\overline{a}, \overline{b}) \in (\mathbb{F}_q^\times)^2 : b^6 = 0\} = 0$ we obtain,

$$I_{\Delta_9}(s, (a, b)) = \begin{cases} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \overline{\chi}(\overline{b}^6), & \textit{if } \overline{\chi} = \chi_{triv} \\ \\ 0 & \textit{all other cases}, \end{cases}$$

where $\overline{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}_q$, we thus get

$$I_{\Delta_9}(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if } \chi=\chi_{triv} \\ \\ (q-1)^2 & \textit{if } \overline{\chi}^6=\chi_{triv} \\ \\ 0 & \textit{all other cases.} \end{cases}$$

Finally, since that $\overline{\chi}^6=\chi_{triv}$ and $\overline{\chi}|_U=\chi_{triv}, U=1+\mathfrak{p}O_K$ is equivalent to $\chi^6=\chi_{triv}$, we obtain

$$Z(s,f,\chi,\Delta_9) = \begin{cases} q^{-1}(1-q^{-1}) & \textit{if } \chi=\chi_{triv}, \\ \\ q^{-1}(1-q^{-1}) & \textit{if } \chi^6=\chi_{triv}, \chi|_U=\chi_{triv} \\ \\ 0 & \textit{all other cases,} \end{cases}$$

A.2 Computation of $Z(s,f,\chi,\Delta_5)$

(An integral on a degenerate face in the sense Kouchnirenko).

$$Z(s,f,\chi,\Delta_5) = \sum_{n=1}^{\infty} \int_{\mathfrak{p}^{3n}O_K^\times \times \mathfrak{p}^{2n}O_K^\times} \chi(ac\ f(x,y))|f(x,y)|^s|dxdy|,$$

$$= \sum_{n=1}^{\infty} q^{-5n-12ns} \int_{O_K^{\times 2}} \chi(ac((y^3-x^2)^2+\mathfrak{p}^{8n}x^4y^4)|(y^3-x^2)^2+\mathfrak{p}^{8n}x^4y^4|^s|dxdy|.$$

Let $f^{(n)}(x,y)=(y^3-x^2)^2+\mathfrak{p}^{8n}x^4y^4$, for $n\geqslant 1$. For compute the integral, $I(s,f^{(n)},\chi)=\int_{O_K^{\times 2}}\chi(ac((y^3-x^2)^2+\mathfrak{p}^{8n}x^4y^4))|(y^3-x^2)^2+\mathfrak{p}^{8n}x^4y^4|^s|dxdy|, n\geqslant 1$, we use the following change of variables:

$$\Phi: \begin{array}{ll} O_K^{\times 2} & \to O_K^{\times 2} \\ (x,y) & \longmapsto (x^3y, x^2y) \end{array}$$

The map Φ gives an analytic bijection of $O_K^{\times 2}$ onto itself and preserves the Haar measure since that its Jacobian $J_\Phi(x,y) = x^4y$ satisfies $|J_\Phi(x,y)|_K = 1$, for every $x, y \in O_K^\times$. Thus
$f^{(n)} \circ \Phi(x,y) = x^{12}y^4\widetilde{f^{(n)}}(x,y)$, with

$$\widetilde{f^{(n)}}(x,y) = (y-1)^2 + \mathfrak{p}^{8n}x^8y^4, \tag{A.2.1}$$

then we have that,

$$I(s, f^{(n)}, \chi) = \int\limits_{O_K^{\times 2}} |\chi(ac(x^{12}y^4\widetilde{f^{(n)}}(x,y))|\widetilde{f^{(n)}}(x,y)|^s|dxdy|.$$

In order to compute the integral $I(s, f^{(n)}, \chi), n \geqslant 1$, we decompose $O_K^{\times 2}$ as follows:

$$O_K^{\times 2} = \bigsqcup_{y_0 \not\equiv 1 (mod \mathfrak{p})} O_K^\times \times \{y_0 + \mathfrak{p}O_K\} \bigcup \left(O_K^\times \times \{1 + \mathfrak{p}O_K\}\right), \tag{A.2.2}$$

where y_0 runs through a set of representatives of $\mathbb{F}_q^\times$ in O_K. From partition (A.2.1) and formula (A.2.2), it follows that,

$$I(s, f^{(n)}, \chi) =$$
$$\sum_{y_0 \not\equiv 1 (mod \mathfrak{p})} \int_{O_K^\times \times \{y_0 + \mathfrak{p}O_K\}} \chi[ac(x^{12}y^4((y-1)^2 + \mathfrak{p}^{8n}x^8y^4))]|(y-1)^2 + \mathfrak{p}^{8n}x^8y^4|^s|dxdy|$$
$$+ \int\limits_{O_K^\times \times \{1 + \mathfrak{p}O_K\}} \chi[ac(x^{12}y^4((y-1)^2 + \mathfrak{p}^{8n}x^8y^4))]|(y-1)^2 + \mathfrak{p}^{8n}x^8y^4|^s|dxdy|.$$

This integral admits the following expansion:

$$I(s,f^{(n)},\chi)=$$
$$\sum_{y_0\not\equiv 1(mod\mathfrak{p})}\sum_{j=0}^{\infty}q^{-1-j}\int_{O_K^\times\times O_K^\times}\mathcal{X}_1(x,y)\ |dxdy|+\sum_{j=0}^{\infty}q^{-1-j}\int_{O_K^\times\times O_K^\times}\mathcal{X}_2(x,y)\ |dxdy|.$$

where

$$\mathcal{X}_1(x,y)=\chi[ac(x^{12}(y_0+\mathfrak{p}^{j+1}y)^4((y_0-1+\mathfrak{p}^{j+1}y)^2+\mathfrak{p}^{8n}x^8(y_0+\mathfrak{p}^{j+1}y)^4))]$$

$$\mathcal{X}_2(x,y)=$$
$$\chi[ac(x^{12}(1+\mathfrak{p}^{j+1}y)^4((\mathfrak{p}^{j+1}y)^2+\mathfrak{p}^{8n}x^8(1+\mathfrak{p}^{j+1}y)^4))]\times|(\mathfrak{p}^{j+1}y)^2+\mathfrak{p}^{8n}x^8(1+\mathfrak{p}^{j+1}y)^4|^s$$

In order to compute integral I, we write $I(s,f^{(n)},\chi)=J_1(s,f^{(n)},\chi)+J_2(s,f^{(n)},\chi)$, where

$$J_1(s,f^{(n)},\chi)=\sum_{y_0\not\equiv 1(mod\mathfrak{p})}\sum_{j=0}^{\infty}q^{-1-j}\int_{O_K^\times\times O_K^\times}\mathcal{X}_1(x,y)\ |dxdy|,$$

and

$$J_2(s,f^{(n)},\chi)=\sum_{j=0}^{\infty}q^{-1-j}\int_{O_K^\times\times O_K^\times}\mathcal{X}_2(x,y)\ |dxdy|.$$

Now, integral $J_2(s,f^{(n)},\chi)$ can write as

$$J_2(s, f^{(n)}, \chi) =$$

$$\sum_{j=0}^{4n-2} q^{-1-j-(2+2j)s} \int\limits_{(O_K^\times)^2} \chi[ac(x^{12}(1+\mathfrak{p}^{j+1}y)^4(y^2+\mathfrak{p}^{8n-(2+2j)}x^8(1+\mathfrak{p}^{j+1}y)^4))]|dxdy|$$
$$+q^{-4n-8ns} \int\limits_{(O_K^\times)^2} \chi[ac(x^{12}(1+\mathfrak{p}^{j+1}y)^4(y^2+x^8(1+\mathfrak{p}^{j+1}y)^4))]|y^2+x^8(1+\mathfrak{p}^{j+1}y)^4|^s|dxdy|$$
$$+\sum_{j=4n}^{\infty} q^{-j-1-8ns} \int\limits_{(O_K^\times)^2} \chi[ac(x^{12}(1+\mathfrak{p}^{j+1}y)^4(\mathfrak{p}^{2+2j-8n}y^2+x^8(1+\mathfrak{p}^{j+1}y)^4))]|dxdy|.$$

Now we obtain,

$$I(s, f^{(n)}, \chi) = \sum_{y_0 \not\equiv 1 (mod \mathfrak{p})} \sum_{j=0}^{\infty} q^{-1-j} \int\limits_{O_K^\times \times O_K^\times} \chi[ac(f_1(x,y))]|dxdy|$$
$$+\sum_{j=0}^{4n-2} q^{-1-j-(2+2j)s} \int\limits_{(O_K^\times)^2} \chi[ac(f_2(x,y))]|dxdy|$$
$$+q^{-4n-8ns} \int\limits_{(O_K^\times)^2} \chi[ac(f_3(x,y))]|f_3(x,y)|_K^s|dxdy|$$
$$+\sum_{j=4n}^{\infty} q^{-j-1-8ns} \int\limits_{(O_K^\times)^2} \chi[ac(f_4(x,y))]|dxdy|,$$

where

$$f_1(x,y) = x^{12}(y_0 + \mathfrak{p}^{j+1}y)^4((y_0 - 1 + \mathfrak{p}^{j+1}y)^2 + \mathfrak{p}^{8n}x^8(y_0 + \mathfrak{p}^{j+1}y)^4),$$
$$f_2(x,y) = x^{12}(1 + \mathfrak{p}^{j+1}y)^4(y^2 + \mathfrak{p}^{8n-(2+2j)}x^8(1 + \mathfrak{p}^{j+1}y)^4),$$
$$f_3(x,y) = x^{12}(1 + \mathfrak{p}^{j+1}y)^4(y^2 + x^8(1 + \mathfrak{p}^{j+1}y)^4),$$
$$f_4(x,y) = x^{12}(1 + \mathfrak{p}^{j+1}y)^4(\mathfrak{p}^{2+2j-8n}y^2 + x^8(1 + \mathfrak{p}^{j+1}y)^4),$$

Now we write, $I(s, f^{(n)}, \chi) = I_1(s, f^{(n)}, \chi) + I_2(s, f^{(n)}, \chi) + I_3(s, f^{(n)}, \chi) + I_4(s, f^{(n)}, \chi)$ with,

$$I_1(s, f^{(n)}, \chi) = \sum_{y_0 \not\equiv 1 (mod \mathfrak{p})} \sum_{j=0}^{\infty} q^{-1-j} \int_{O_K^\times \times O_K^\times} \chi[ac(f_1(x,y))]|dxdy|.$$
$$I_2(s, f^{(n)}, \chi) = \sum_{j=0}^{4n-2} q^{-1-j-(2+2j)s} \int_{(O_K^\times)^2} \chi[ac(f_2(x,y))]|dxdy|.$$
$$I_3 = (s, f^{(n)}, \chi) q^{-4n-8ns} \int_{(O_K^\times)^2} \chi[ac(f_3(x,y))]|f_3(x,y)|^s|dxdy|.$$
$$I_4 = (s, f^{(n)}, \chi) \sum_{j=4n}^{\infty} q^{-j-1-8ns} \int_{(O_K^\times)^2} \chi[ac(f_4(x,y))]|dxdy|.$$

And we find every integral $I_i(s, f^{(n)}, \chi), i = 1, 2, 3, 4$ after we compute

$$Z(s, f, \chi, \Delta_5) = \sum_{n=1}^{\infty} q^{-5n-12ns} I(s, f^{(n)}, \chi).$$

(a) $I_1(s, f^{(n)}, \chi) = \sum_{y_0 \not\equiv 1 (mod \mathfrak{p})} \sum_{j=0}^{\infty} q^{-1-j} \int_{O_K^\times \times O_K^\times} \chi(ac(f_1(x,y))) \ |dxdy|.$

Since polynomial

$$f_1(x,y) = x^{12}(y_0 + \mathfrak{p}^{j+1}y)^4((y_0 - 1 + \mathfrak{p}^{j+1}y)^2 + \mathfrak{p}^{8n}x^8(y_0 + \mathfrak{p}^{j+1}y)^4),$$

we have $\overline{f_1}(x,y)=x^{12}y_0^4(y_0-1)^2$.

By using equation (A.1.1), so we obtain that,

$$I_1(s,f^{(n)},\chi)=\sum_{y_0\not\equiv 1(mod\mathfrak{p})}\sum_{j=0}^{\infty}q^{-1-j}\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2}\int_{(a,b)+(\mathfrak{p}O_K)^2}\chi(ac\ f_1(x,y))|dxdy|,$$
$$=\sum_{y_0\not\equiv 1(\mod \mathfrak{p})}\sum_{j=0}^{\infty}q^{-3-j}\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2}\int_{O_K^2}\chi(ac\ f_1(a+\mathfrak{p}x,b+\mathfrak{p}y))|dxdy|.$$

Then we apply the change variables (A.1.3) to function f_1, and we note that

$$\frac{\partial\overline{f_1}}{\partial x}(\overline{a},\overline{b})=12\overline{y}_0^4(\overline{y}_0-1)^2\overline{a}^{11}\neq 0,$$

then

$$I_1(s,f^{(n)},\chi)=\sum_{y_0\not\equiv 1(\mod \mathfrak{p})}\sum_{j=0}^{\infty}q^{-3-j}\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2}\int_{O_K}\chi(ac\ (f_1(a,b)+\mathfrak{p}z_1))|dz_1|,$$
$$=\sum_{y_0\not\equiv 1(\mod \mathfrak{p})}\sum_{j=0}^{\infty}q^{-3-j}\overline{I_1}(s,(a,b)),$$

where $\overline{I_1}(s,(a,b))=\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2}\int_{O_K}\chi(ac\ (f_1(a,b)+\mathfrak{p}z_1))|dz_1|$, for to compute it we use the Lemma 1.2.2, and given that $Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2:\overline{a}^{12}\overline{y}_0^4(\overline{y}_0-1)^2=0\}=0$, we get

$$\overline{I_1}(s,(a,b))=\begin{cases}(q-1)^2 & if\ \ \chi=\chi_{triv}\\ \\ \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2}\overline{\chi}(\overline{a}^{12}\overline{y}_0^4(\overline{y}_0-1)^2) & if\ \ \overline{\chi}=\chi_{triv}\\ \\ 0 & all\ other\ cases,\end{cases}$$

where $\overline{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}_q$, then we have that

$$\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \overline{\chi}(\overline{a}^{12}\overline{y}_0^4(\overline{y}_0-1)^2) = \begin{cases} \overline{\chi}^4(\overline{y}_0)\overline{\chi}^2(\overline{y}_0-1)(q-1)^2, & if\ \overline{\chi}^{12}=1 \\ (q-1)\cdot 0 = 0 & if\ \overline{\chi}^{12}\neq 1, \end{cases}$$

Thus,

$$\overline{I_1}(s,(a,b)) = \begin{cases} (q-1)^2 & if\ \chi=\chi_{triv} \\ \overline{\chi}^4(\overline{y}_0)\overline{\chi}^2(\overline{y}_0-1)(q-1)^2 & if\ \overline{\chi}^{12}=\chi_{triv} \\ 0 & all\ other\ cases. \end{cases}$$

Finally, since that $\overline{\chi}^{12} = \chi_{triv}$ and $\chi|_U = \chi_{triv}, U = 1 + \mathfrak{p}O_K$ is equivalent to $\chi^{12} = \chi_{triv}$, and furthermore

$I_1(s, f^{(n)}, \chi) = \sum_{y_0\not\equiv 1(\mod \mathfrak{p})} \sum_{j=0}^{\infty} q^{-3-j}\overline{I_1}(s,(a,b))$, we obtain

$$I_1(s,f^{(n)},\chi) = \begin{cases} q^{-1}(1-q^{-1})(q-2) & if\ \chi=\chi_{triv} \\ \overline{\chi}^4(\overline{y}_0)\overline{\chi}^2(\overline{y}_0-1)q^{-1}(1-q^{-1})(q-2) & if\ \chi^{12}=\chi_{triv}, \chi|_U=\chi_{triv} \\ 0 & all\ other\ cases, \end{cases}$$

(b) $I_2(s, f^{(n)}, \chi) = \sum_{j=0}^{4n-2} q^{-1-j-(2+2j)s} \int_{(O_K^\times)^2} \chi[ac(f_2(x,y))]|dxdy|$.

Since polynomial $f_2(x,y) = x^{12}(1+\mathfrak{p}^{j+1}y)^4(y^2+\mathfrak{p}^{8n-(2+2j)}x^8(1+\mathfrak{p}^{j+1}y)^4)$, we have $\overline{f_2}(x,y) = x^{12}y^2$.

By using equation (A.1.1) so we obtain that,

$$I_2(s, f^{(n)}, \chi) = \sum_{j=0}^{4n-2} q^{-1-j-(2+2j)s} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{(a,b)+(\mathfrak{p}O_K)^2} \chi(ac\ f_2(x,y))|dxdy|,$$
$$= \sum_{j=0}^{4n-2} q^{-3-j-(2+2j)s} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K^2} \chi(ac\ f_2(a+\mathfrak{p}x, b+\mathfrak{p}y))|dxdy|.$$

Then we apply the change variables (A.1.3) to function f_2 where

$$\frac{\partial \overline{f_2}}{\partial x}(\overline{a}, \overline{b}) = 12(\overline{a}^{11}\overline{b}^2) \neq 0,$$

we use the change of variables above and we obtain that,

$$I_2(s, f^{(n)}, \chi) = \sum_{j=0}^{4n-2} q^{-3-j-(2+2j)s} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K} \chi(ac\ (f_2(a,b) + \mathfrak{p}z_1))|dz_1|$$
$$= \sum_{j=0}^{4n-2} q^{-3-j-(2+2j)s}\overline{I}_2(s, (a,b)),$$

where $\overline{I}_2(s,(a,b)) = \sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}} \int_{O_K} \chi(ac\ (f_2(a,b) + \mathfrak{p}z_1))|dz_1|$, given that

$$N = Card\{(\overline{a}, \overline{b}) \in (\mathbb{F}_q^{\times})^2 : \overline{a}^{12}\overline{b}^2 = 0\} = 0,$$

we can assert that

$$\overline{I_2}(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if } \chi = \chi_{triv} \\ \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \overline{\chi}^{12}(\overline{a})\overline{\chi}^2(\overline{b}) & \textit{if } \overline{\chi} = \chi_{triv} \\ 0 & \textit{all other cases}, \end{cases}$$

where $\overline{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}_q$. Then we conclude

$$\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \overline{\chi}^{12}(\overline{a})\overline{\chi}^2(\overline{b}) = \begin{cases} (q-1)^2 & \textit{if } \overline{\chi}^2 = \chi_{triv} \\ 0 & \textit{if } \overline{\chi}^2 \neq \chi_{triv} \end{cases}$$

Thus, $$\overline{I_2}(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if } \chi = \chi_{triv} \\ (q-1)^2 & \textit{if } \overline{\chi}^2 = \chi_{triv} \\ 0 & \textit{all other cases.} \end{cases}$$

Finally, since that $\overline{\chi}^2 = \chi_{triv}$ and $\chi|_U = \chi_{triv}$, $U = 1 + \mathfrak{p}O_K$ is equivalent to $\chi^2 = \chi_{triv}$ and the identity $\sum_{k=A}^{B} z^k = \frac{z^A - z^{B+1}}{1-z}$, we obtain that

$$I_2(s, f^{(n)}, \chi) = \begin{cases} \frac{q^{-1-2s}(1-q^{(4n-1)(-1-2s)})(1-q^{-1})^2}{1-q^{-1-2s}} & \textit{if } \chi = \chi_{triv}, \\ \frac{q^{-1-2s}(1-q^{(4n-1)(-1-2s)})(1-q^{-1})^2}{1-q^{-1-2s}} & \textit{if } \chi^2 = \chi_{triv}, \chi|_U = \chi_{triv} \\ 0 & \textit{all other cases,} \end{cases}$$

(c) $I_3(s, f^{(n)}, \chi) = q^{-4n-8ns} \int\limits_{(O_K^\times)^2} \chi(ac(f_3(x,y)))\ |f_3(x,y)|^s|\ \ dxdy|.$

Since $f_3(x,y) = x^{12}(1+\mathfrak{p}^{j+1}y)^4(y^2 + x^8(1+\mathfrak{p}^{j+1}y)^4)$, we have $\overline{f_3}(x,y) = x^{12}y^2 + x^{20}$.

By using equation (A.1.1), so we obtain that,

$$I_3(s, f^{(n)}, \chi) = q^{-4n-8ns} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int\limits_{(a,b)+(\mathfrak{p}O_K)^2} \chi(ac\ f_3(x,y))|f_3(x,y)|^s|dxdy|,$$

$$= q^{-4n-8ns-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int\limits_{O_K^2} \chi(ac\ f_3(a+\mathfrak{p}x, b+\mathfrak{p}y))|f_3(a+\mathfrak{p}x, b+\mathfrak{p}y)|^s|dxdy|.$$

Then we apply the change variables (A.1.3) to function f_3 where $\frac{\partial \overline{f_3}}{\partial y}(\overline{a},\overline{b}) = 2(\overline{a}^{12}\overline{b}) \neq 0$, and we obtain

$$\begin{aligned} I_3(s, f^{(n)}, \chi) =& q^{-4n-8ns-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K} \chi(ac\ (f_3(a,b)+\mathfrak{p}z_1))|f_3(a,b)+\mathfrak{p}z_1)|^s|dz_1| \\ =& q^{-4n-8ns-2}\overline{I_3}(s,(a,b)), \end{aligned}$$

where $\overline{I}_3(s,(a,b)) = \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K} \chi(ac\,(f_3(a,b)+\mathfrak{p}z_1))|f_3(a,b)+\mathfrak{p}z_1)|^s|dz_1|$, thus we can resolve it applying the Lemma 1.2.2, and we obtain,

$$\overline{I}_3(s,(a,b)) = I_{3,1}(s,(a,b)) + I_{3,2}(s,(a,b)),$$

where

$$I_{3,1}(s,(a,b)) = \begin{cases} \frac{q^{-s}(1-q^{-1})N}{(1-q^{-1-s})} + (q-1)^2 - N & \textit{if } \chi = \chi_{triv} \\ \\ 0 & \textit{in other case}, \end{cases}$$

where

$$\begin{aligned} N =& (q-1)Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2 : \overline{f}_3(\overline{a},\overline{b}) = 0\}, \\ =& Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2 : \overline{a}^{12}(\overline{b}^2 + \overline{a}^8) = 0\} = (q-1)Card\{x \in \mathbb{F}_q^{\times} : x^2 = -1\}. \end{aligned}$$

On the other hand

$$I_{3,2}(s,(a,b)) = \begin{cases} \sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2 \\ f_3(\overline{a},\overline{b})\neq 0}} \chi(ac(f_3(\overline{a},\overline{b}))) & \textit{if } \chi|_U = \chi_{triv} \\ \\ 0 & \textit{in other case}, \end{cases}$$

where $U = 1 + \mathfrak{p}O_K$.

Now, since that $\overline{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}_q$, we have that

$$I_{3,2}(s,(a,b)) = \begin{cases} \sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2 \\ (\overline{b}^2+\overline{a}^8)\neq 0}} \overline{\chi}(\overline{a}^{12}(\overline{b}^2+\overline{a}^8)) & \text{if } \overline{\chi}=\chi_{triv} \\ \\ 0 & \text{all other cases.} \end{cases}$$

Now since that $\overline{\chi}=\chi_{triv}$ and $\chi|_U=\chi_{triv}$ implies $\chi=\chi_{triv}$ we get

$$I_{3,2}(s,(a,b)) = \begin{cases} \sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2 \\ (\overline{b}^2+\overline{a}^8)\neq 0}} \chi^{12}(\overline{a})\chi(\overline{b}^2+\overline{a}^8) & \text{if } \chi=\chi_{triv} \\ \\ 0 & \text{all other cases.} \end{cases}$$

Thus we can write

$$I_{3,2}(s,(a,b)) = \begin{cases} T & \text{if } \chi=\chi_{triv} \\ \\ 0 & \text{all other cases,} \end{cases}$$

where $T=\sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2 \\ (\overline{b}^2+\overline{a}^8)\neq 0}} \chi^{12}(\overline{a})\chi(\overline{b}^2+\overline{a}^8)$.

Finally, since that $I_3(s,(a,b))=q^{-4n-8ns-2}\overline{I}_3(s,(a,b))$, we obtain that

$I_3=q^{-4n-8ns-2}(I_{3,1}(s,(a,b))+I_{3,2}(s,(a,b)))$, and therefore

$$I_3 = \begin{cases} q^{-4n-8ns-2}\left(\frac{q^{-s}(1-q^{-1})N}{(1-q^{-1-s})}+(q-1)^2-N+T\right) & \text{if } \chi=\chi_{triv} \\ \\ 0 & \text{in other case,} \end{cases}$$

(d) $I_4=\sum_{j=4n}^{\infty} q^{-j-1-8ns} \int_{(O_K^\times)^2} \chi(ac(f_4(x,y)))\ |dxdy|$

Since polynomial $f_4(x,y)=x^{12}(1+\mathfrak{p}^{j+1}y)^4(\mathfrak{p}^{2+2j-8n}y^2+x^8(1+\mathfrak{p}^{j+1}y)^4)$, we have $\overline{f_4}(x,y)=x^{20}$.

By using equation (A.1.1), so we obtain that,

$$\begin{aligned} I_4 &= \sum_{j=4n}^{\infty} q^{-j-1-8ns} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int_{(a,b)+(\mathfrak{p}O_K)^2} \chi(ac\ f_4(x,y))|dxdy|, \\ &= \sum_{j=4n}^{\infty} q^{-j-3-8ns} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int_{O_K^2} \chi(ac\ f_4(a+\mathfrak{p}x,b+\mathfrak{p}y))|dxdy|. \end{aligned}$$

By applying the change variables (A.1.3) to function f_4 and since that $\frac{\partial \overline{f_4}}{\partial x}(\overline{a},\overline{b}) = 20\overline{a}^{19} \neq 0$, then

$$I_4 = \sum_{j=4n}^{\infty} q^{-j-8ns-3} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K} \chi(ac\ (f_4(a,b)+\mathfrak{p}z_1))|dz_1|,$$
$$= \sum_{j=4n}^{\infty} q^{-j-8ns-3}\overline{I}_4(s,(a,b)),$$

where $\overline{I}_4(s,(a,b)) = \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K} \chi(ac\ (f_4(a,b)+\mathfrak{p}z_1))|dz_1|$ for to compute it we use the Lemma 1.2.2, and given that $N = Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2 : \overline{a}^{20} = 0\} = 0$, we get

$$\overline{I_4}(s,(a,b)) = \begin{cases} (q-1)^2 & if\ \ \chi = \chi_{triv} \\ \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \overline{\chi}(\overline{a}^{20}) & if\ \overline{\chi} = \chi_{triv} \\ 0 & all\ other\ cases, \end{cases}$$

where $\overline{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}_q$, we deduce,

$$\sum_{\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \overline{\chi}(\overline{a}^{20}) = \begin{cases} (q-1)^2, & if\ \ \overline{\chi}^{20} = \chi_{triv} \\ 0 & if\ \ \overline{\chi}^{20} \neq \chi_{triv}. \end{cases}$$

Then we have that, $\overline{I_4}(s,(a,b)) = \begin{cases} (q-1)^2 & if\ \ \chi = \chi_{triv} \\ (q-1)^2 & if\ \ \overline{\chi}^{20} = \chi_{triv} \\ 0 & all\ other\ cases. \end{cases}$

Finally, since that $\overline{\chi}^{20} = \chi_{triv}$ and $\chi|_U = \chi_{triv}$ is equivalent to $\chi^{20} = \chi_{triv}$ and $I_4 = \sum_{j=4n}^{\infty} q^{-j-8ns-3}\overline{I}_4(s,(a,b))$, we can assert that

$$I_4=\begin{cases} q^{-4n-8ns-1}(1-q^{-1}) & \textit{if } \chi=\chi_{triv}, \\ q^{-4n-8ns-1}(1-q^{-1}) & \textit{if } \chi^{20}=\chi_{triv}, \chi|_U=\chi_{triv} \\ 0 & \textit{all other cases.} \end{cases}$$

Now, since that $Z(s,f,\chi,\Delta_5)=\sum_{n=1}^{\infty}q^{-5n-12ns}I=\sum_{n=1}^{\infty}q^{-5n-12ns}\sum_i I_i$, for $i=1,\cdots,4$, then

When $\chi=\chi_{triv}$,

$$\begin{aligned} Z(s,g,\chi,\Delta_5)= \\ \frac{(1-q^{-1})^2q^{-6-14s}}{(1-q^{-1-2s})(1-q^{-5-12s})}-\frac{(1-q^{-1})^2q^{-9-20s}}{(1-q^{-1-2s})(1-q^{-9-20s})} \\ +\frac{(q-2)(1-q^{-1})q^{-6-12s}}{(1-q^{-5-12s})}+\frac{(1-q^{-1})(q^{-10-20s})}{(1-q^{-9-20s})} \\ +\frac{q^{-9-20s}}{(1-q^{-1-s})(1-q^{-9-20s})}\{q^{-1}(q^{-1-s}-q^{-1})N+(1-q^{-1})^2(1-q^{-1-s}) \\ -q^{-2}(1-q^{-1-s})T\}, \end{aligned} \tag{A.2.3}$$

where $N=(q-1)Card\{x\in\mathbb{F}_q^{\times}:x^2=-1\}$ and $T=Card\{(x,y)\in(\mathbb{F}_q^{\times})^2|y^2+x^8=0\}$.

When $\chi\neq\chi_{triv}$ and $\chi|_{1+\mathfrak{p}O_K}=\chi_{triv}$ we several cases: if $\chi^2=\chi_{triv}$, we have

$$\begin{aligned} Z(s,f,\chi,\Delta_5) &= \sum_{n=1}^{\infty}q^{-5n-12ns}\frac{(1-q^{-1})^2q^{-1-2s}(1-q^{(4n-1)(-1-2s)})}{(1-q^{-1-2s})} \\ &= \frac{(1-q^{-1})^2q^{-6-14s}}{(1-q^{-1-2s})(1-q^{-5-12s})}-\frac{(1-q^{-1})^2q^{-9-20s}}{(1-q^{-1-2s})(1-q^{-9-20s})}. \end{aligned} \tag{A.2.4}$$

If $\chi^{12}=\chi_{triv}$, then

$$\begin{aligned}Z(s,f,\chi,\Delta_5) &= \overline{\chi}^4(\overline{y}_0)\overline{\chi}^2(\overline{y}_0-1)\sum_{n=1}^{\infty}q^{-5n-12ns}(1-q^{-1})(q-2)q^{-1}\\ &= \overline{\chi}^4(\overline{y}_0)\overline{\chi}^2(\overline{y}_0-1)\frac{(q-2)(1-q^{-1})q^{-6-12s}}{(1-q^{-5-12s})}. \qquad \text{(A.2.5)}\end{aligned}$$

For $\chi^{20}=\chi_{triv}$,

$$\begin{aligned}Z(s,f,\chi,\Delta_5) &= \sum_{n=1}^{\infty}q^{-5n-12ns}(1-q^{-1})(q^{-4n-8ns-1})\\ &= \frac{(1-q^{-1})(q^{-10-20s})}{(1-q^{-9-20s})}. \qquad \text{(A.2.6)}\end{aligned}$$

In all other cases, $Z(s,f,\chi,\Delta_5)=0$.

Summarizing the result obtain for all cones,

For $\chi=\chi_{triv}$,

$$\begin{aligned}Z(s,f,\chi_{triv}) &= 2q^{-1}(1-q^{-1})+\frac{q^{-2-4s}(1-q^{-1})}{(1-q^{-2-4s})}+\frac{q^{-7-16s}(1-q^{-1})^2}{(1-q^{-2-4s})(1-q^{-5-12s})}\\ &+\frac{q^{-8-18s}(1-q^{-1})^2}{(1-q^{-3-6s})(1-q^{-5-12s})}+\frac{q^{-3-6s}(1-q^{-1})}{(1-q^{-3-6s})}+\frac{(1-q^{-1})^2q^{-6-14s}}{(1-q^{-1-2s})(1-q^{-5-12s})}\\ &-\frac{(1-q^{-1})^2q^{-9-20s}}{(1-q^{-1-2s})(1-q^{-9-20s})}+\frac{(q-2)(1-q^{-1})q^{-6-12s}}{(1-q^{-5-12s})}+\frac{(1-q^{-1})(q^{-10-20s})}{(1-q^{-9-20s})}\\ &+\frac{q^{-9-20s}}{(1-q^{-1-s})(1-q^{-9-20s})}\{q^{-1}(q^{-1-s}-q^{-1})N+(1-q^{-1})^2(1-q^{-1-s})\\ &\qquad -q^{-2}(1-q^{-1-s})T\}, \qquad \text{(A.2.7)}\end{aligned}$$

where $N = (q-1)Card\{x \in \mathbb{F}_q^\times : x^2 = -1\}$ and $T = Card\{(x,y) \in (\mathbb{F}_q^\times)^2 | y^2 + x^8 = 0\}$.

When $\chi \neq \chi_{triv}$ and $\chi|_{1+\mathfrak{p}O_K} = \chi_{triv}$ we several cases: if $\chi^2 = \chi_{triv}$, we have

$$\begin{aligned} Z(s,f,\chi) &= \sum_{n=1}^{\infty} q^{-5n-12ns} \frac{(1-q^{-1})^2 q^{-1-2s}(1-q^{(4n-1)(-1-2s)})}{(1-q^{-1-2s})} \\ &= \frac{(1-q^{-1})^2 q^{-6-14s}}{(1-q^{-1-2s})(1-q^{-5-12s})} - \frac{(1-q^{-1})^2 q^{-9-20s}}{(1-q^{-1-2s})(1-q^{-9-20s})}. \end{aligned} \tag{A.2.8}$$

When $\chi^4 = \chi_{triv}$,

$$\begin{aligned} Z(s,f,\chi) = q^{-1}(1-q^{-1}) + \frac{q^{-3-4s}(1-q^{-1})}{(1-q^{-2-4s})} + \frac{q^{-2-4s}(1-q^{-1})^2}{(1-q^{-2-4s})} \\ + \frac{q^{-7-16s}(1-q^{-1})^2}{(1-q^{-2-4s})(1-q^{-5-12s})}. \end{aligned} \tag{A.2.9}$$

$\chi^6 = \chi_{triv}$, we obtain

$$\begin{aligned} Z(s,f,\chi) = \frac{q^{-8-18s}(1-q^{-1})^2}{(1-q^{-3-6s})(1-q^{-5-12s})} + \frac{q^{-3-6s}(1-q^{-1})^2}{(1-q^{-3-6s})} \\ + \frac{q^{-4-6s}(1-q^{-1})}{(1-q^{-3-6s})} + q^{-1}(1-q^{-1}). \end{aligned} \tag{A.2.10}$$

For $\chi^{12} = \chi_{triv}$, then

$$\begin{aligned} Z(s,f,\chi) &= \overline{\chi}^4(\overline{y}_0)\overline{\chi}^2(\overline{y}_0 - 1)\sum_{n=1}^{\infty} q^{-5n-12ns}(1-q^{-1})(q-2)q^{-1} \\ &= \overline{\chi}^4(\overline{y}_0)\overline{\chi}^2(\overline{y}_0 - 1)\frac{(q-2)(1-q^{-1})q^{-6-12s}}{(1-q^{-5-12s})}, \end{aligned} \tag{A.2.11}$$

where $\overline{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}_q^{\times}$. Finally for $\chi^{20} = \chi_{triv}$,

$$\begin{aligned} Z(s,f,\chi) &= \sum_{n=1}^{\infty} q^{-5n-12ns}(1-q^{-1})(q^{-4n-8ns-1}) \\ &= \frac{(1-q^{-1})(q^{-10-20s})}{(1-q^{-9-20s})}. \end{aligned} \tag{A.2.12}$$

In all other cases, $\sum Z(s,f,\chi,\Delta_i) = 0$.

Appendix B

The local zeta function of $(y^3 - x^2)^2(y^3 - cx^2) + x^4y^4$

In this section we present the example 2.3.2 computed in full detail. In this example we assume that the characteristic of the residue field of K is different from 2 and 3. We shall compute explicitly the local zeta functions for $g(x, y) = (y^3 - x^2)^2(y^3 - cx^2) + x^4y^4 \in K[x, y]$, with $c \in O_K^\times$ and $c \not\equiv 1(mod\ \mathfrak{p})$. This polynomial is degenerate with respect to its geometric Newton polygon in the sense of Kouchnirenko. We obtain an explicit list of candidates for the poles in terms of geometric data obtained from a family of arithmetic Newton polygons attached to the polynomial $g(x, y)$.

The conical subdivision of $\mathbb{R}_+^2$ subordinated to the geometric Newton polygon of $g(x, y)$ is $\mathbb{R}_+^2 = \{(0, 0) \cup \bigcup_{j=1}^{9} \Delta_j\}$, and it do possible to reduce the computation of $Z(s, g, \chi)$ to the computation of the $p-$ adic integrals $Z(s, g, \chi, O_K^\times), Z(s, g, \chi, \Delta_i),\ i = 1, \cdots, 9$.

B.1 Computation of $Z(s, g, \chi, \Delta_i)$

These integrals correspond to the case in which g is non-degenerate on Δ_i.

(a) Case $Z(s, g, \chi, \Delta_1)$.

$$Z(s, g, \chi, \Delta_1) = \sum_{n=1}^{\infty} \int_{O_K^{\times} \times \mathfrak{p}^n O_K^{\times}} \chi(ac\ g(x, y))|g(x, y)|^s |dxdy|,$$
$$= \sum_{n=1}^{\infty} q^{-n} \int_{O_K^{\times 2}} \chi(ac\ (g_1(x, y)))\ |g_1(x, y)|.$$

where the polynomial $g_1(x, y) = (\mathfrak{p}^{3n}y^3 - x^2)^2(\mathfrak{p}^{3n}y^3 - cx^2) + \mathfrak{p}^{4n}x^4y^4$, and $\overline{g_1}(x, y) = -cx^6$. By using equation (A.1.1), thus

$$Z(s, g, \chi, \Delta_1) = \sum_{n=1}^{\infty} q^{-n} \sum_{(\overline{a}, \overline{b}) \in \mathbb{F}_q^{\times 2}} \int_{(a,b)+(\mathfrak{p}O_K)^2} \chi(ac\ g_1(x, y))|g_1(x, y)|^s |dxdy|,$$
$$= \sum_{n=1}^{\infty} q^{-n-2} \sum_{(\overline{a}, \overline{b}) \in \mathbb{F}_q^{\times 2}} \int_{O_K^2} \chi(ac\ g_1(a + \mathfrak{p}x, b + \mathfrak{p}y))|g_1(a + \mathfrak{p}x, b + \mathfrak{p}y)|^s |dxdy|.$$

Now we apply the change variables (A.1.3) to function g_1 and since that $\frac{\partial \overline{g_1}}{\partial x}(\overline{a}, \overline{b}) = -6c\overline{a}^5 \not\equiv 0 (\mathrm{mod}\ \mathfrak{p})$, then

$$Z(s, g, \chi, \Delta_1) =$$
$$\sum_{n=1}^{\infty} q^{-n-2} \sum_{(\overline{a}, \overline{b}) \in \mathbb{F}_q^{\times 2}} \int_{O_K^2} \chi(ac\ g_1(a + \mathfrak{p}x, b + \mathfrak{p}y))|g_1(a + \mathfrak{p}x, b + \mathfrak{p}y)|^s |dxdy|,$$
$$= \sum_{n=1}^{\infty} q^{-n-2} \sum_{(\overline{a}, \overline{b}) \in \mathbb{F}_q^{\times 2}} \int_{O_K} \chi(ac\ (g_1(a, b) + \mathfrak{p}z_1))|g_1(a, b) + \mathfrak{p}z_1|^s |dz_1|,$$
$$= \sum_{n=1}^{\infty} q^{-n-2} I_{\Delta_1}(s, (a, b)),$$

where $I_{\Delta_1}(s,(a,b)) = \sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}} \int\limits_{O_K} \chi(ac\,(g_1(a,b)+\mathfrak{p}z_1))|g_1(a,b)+\mathfrak{p}z_1|^s|dz_1|$, then by Lemma 1.2.2 and given that

$$N = Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2 : \overline{g}_1(\overline{a},\overline{b})=0\} = Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2 : -c\overline{a}^6=0\} = 0,$$

then we get

$$I_{\Delta_1}(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if}\ \ \chi=\chi_{triv} \\ \sum_{\substack{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}\\ \overline{g}_1(\overline{a},\overline{b})\neq 0}} \overline{\chi}(\overline{g}_1(\overline{a},\overline{b})) & \textit{if}\ \ \overline{\chi}=\chi_{triv} \\ 0 & \textit{all other cases}, \end{cases}$$

where $\overline{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}_q$. Thus,

$$I_{\Delta_1}(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if}\ \ \chi=\chi_{triv} \\ \sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}} \overline{\chi}(-\overline{c}\overline{a}^6) & \textit{if}\ \ \overline{\chi}=\chi_{triv} \\ 0 & \textit{all other cases}, \end{cases}$$

Now since that $\overline{\chi}^6=\chi_{triv}$, and $\chi|_U=\chi_{triv}$, $U=1+\mathfrak{p}O_K$ implies $\chi^6=\chi_{triv}$, we have

$$\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}} \overline{\chi}(-\overline{c}\overline{a}^6) = \begin{cases} \chi(-\overline{c})\cdot 0 = 0 & \textit{if}\ \ \chi^6\neq\chi_{triv}, \\ \chi(-\overline{c})(q-1)^2 & \textit{if}\ \ \chi^6=\chi_{triv}, \end{cases} \tag{B.1.1}$$

Therefore,

$$I_{\Delta_1}(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if}\ \ \chi=\chi_{triv} \\ \chi(-\overline{c})(q-1)^2 & \textit{if}\ \ \chi^6=\chi_{triv},,\chi|_U=\chi_{triv}, \\ 0 & \textit{all other cases.} \end{cases}$$

Finally, since that $Z(s,g,\chi,\Delta_1)=\sum_{n=1}^{\infty} q^{-n-2} I_{\Delta_1}(s,(a,b))$, we conclude

$$Z(s,g,\chi,\Delta_1)=\begin{cases} q^{-1}(1-q^{-1}) & \textit{if } \chi=\chi_{triv} \\ \chi(-\overline{c})q^{-1}(1-q^{-1}), & \textit{if } \chi^6=\chi_{triv}, \chi|_U=\chi_{triv}, \\ 0 & \textit{all other cases}, \end{cases}$$

(b) Case $Z(s,g,\chi,\Delta_2)$.

$$Z(s,g,\chi,\Delta_2)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_{\mathfrak{p}^m O_K^\times\times\mathfrak{p}^{n+m}O_K^\times}\chi(ac\ (g(x,y))|g(x,y)|^s\ |dxdy|,$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{-2m-n-6ms}\int_{O_K^{\times 2}}\chi(ac\ (g_2(x,y)))\ |g_2(x,y)|^s\ |dxdy|.$$

Since that polynomial $g_2(x,y)=(\mathfrak{p}^{3n+m}y^3-x^2)^2(\mathfrak{p}^{3n+m}y^3-cx^2)+\mathfrak{p}^{4n+2m}x^4y^4$, and $\overline{g_2}(x,y)=-cx^6$ thus we obtain that the origin of K is the only singular point of $g_2(x,y)$ over $(\mathbb{F}_q^\times)^2$. By using equation (A.1.1), so we obtain that,

$$Z(s,g,\chi,\Delta_2)=$$
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{-2m-n-6ms}\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\int_{(a,b)+(\mathfrak{p}O_K)^2}\chi(ac\ g_2(x,y))|g_2(x,y)|^s|dxdy|,$$
$$=\sum_{m=n=1}^{\infty}q^{-2m-n-6ms-2}\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\int_{O_K^2}\mathcal{X}(g_2(a+\mathfrak{p}x,b+\mathfrak{p}y))|dxdy|,$$

where $\mathcal{X}(g_2(a+\mathfrak{p}x,b+\mathfrak{p}y)=\chi(ac(g_2(a+\mathfrak{p}x,b+\mathfrak{p}y)))\ |g_2(a+\mathfrak{p}x,b+\mathfrak{p}y)|^s$. Now we apply the change variables (A.1.3) to function g_2 and since that $\frac{\partial\overline{g_2}}{\partial x}(\overline{a},\overline{b})=-6\overline{ca}^5\neq 0$, then

$$Z(s,g,\chi,\Delta_2)=$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{-2m-n-6ms-2}\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\int_{O_K}\chi(ac\ (g_2(a,b)+\mathfrak{p}z_1))|g_2(a,b)+\mathfrak{p}z_1)|^s|dz_1|,$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{-2m-n-6ms-2}I_{\Delta_2}(s,(a,b)),$$

where $I_{\Delta_2}(s,(a,b))=\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\int_{O_K}\chi(ac\ (g_2(a,b)+\mathfrak{p}z_1))|g_2(a,b)+\mathfrak{p}z_1)|^s|dz_1|$.

Then since that $N=Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2:\overline{g}_2(\overline{a},\overline{b})=0\}=Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2:-\overline{ca}^6=0\}=0$, we have

$$I_{\Delta_2}(s,(a,b))=\begin{cases}(q-1)^2 & if\ \ \chi=\chi_{triv}\\ \sum_{\substack{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}\\ \overline{g}_2(\overline{a},\overline{b})\neq 0}}\overline{\chi}(\overline{g}_2(\overline{a},\overline{b})) & if\ \ \overline{\chi}=\chi_{triv}\\ 0 & all\ other\ cases,\end{cases}$$

where $\overline{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}_q$. Then

$$I_{\Delta_2}(s,(a,b))=\begin{cases}(q-1)^2 & if\ \ \chi=\chi_{triv}\\ \sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\overline{\chi}(-\overline{ca}^6) & if\ \ \overline{\chi}=\chi_{triv}\\ 0, & all\ other\ cases.\end{cases}$$

Now since that $\overline{\chi}^6=\chi_{triv}$ and $\chi|_U=\chi_{triv},\ \ U=1+\mathfrak{p}O_K$ implies $\chi^6=\chi_{triv}$, thus follows by the same method as in procedure above,

Finally since $Z(s,g,\chi,\Delta_2)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{-2m-n-6ms-2}I_{\Delta_2}(s,(a,b))$, we obtain

$$Z(s,g,\chi,\Delta_2)=\begin{cases}\frac{(1-q^{-1})q^{-3-6s}}{1-q^{-2-6s}} & \textit{if } \chi=\chi_{triv}\\ \chi(-\overline{c})\frac{q^{-3-6s}(1-q^{-1})}{(1-q^{-2-6s})} & \textit{if } \chi^6=\chi_{triv},\chi|_U=\chi_{triv}\\ 0 & \textit{all other cases,}\end{cases}$$

(c) Case $Z(s,g,\chi,\Delta_3)$.

$$Z(s,g,\chi,\Delta_3)=\sum_{n=1}^{\infty}\int_{\mathfrak{p}^nO_K^\times\times\mathfrak{p}^nO_K^\times}\chi(ac\ g(x,y))|g(x,y)|^s|dxdy|,$$
$$=\sum_{n=1}^{\infty}q^{-2n-6ns}\int_{O_K^{\times 2}}\mathcal{X}(\mathfrak{p}^ny^3-x^2)^2(\mathfrak{p}^ny^3-cx^2)+\mathfrak{p}^{2n}x^4y^4)\ |dxdy|.$$

where

$$\mathcal{X}((\mathfrak{p}^ny^3-x^2)^2(\mathfrak{p}^ny^3-cx^2)+\mathfrak{p}^{2n}x^4y^4)=$$
$$\chi(ac\ (\mathfrak{p}^ny^3-x^2)^2(\mathfrak{p}^ny^3-cx^2)+\mathfrak{p}^{2n}x^4y^4)|(\mathfrak{p}^ny^3-x^2)^2(\mathfrak{p}^ny^3-cx^2)+\mathfrak{p}^{2n}x^4y^4|^s$$

Since that polynomial $g_3(x,y)=(\mathfrak{p}^ny^3-x^2)^2(\mathfrak{p}^ny^3-cx^2)+\mathfrak{p}^{2n}x^4y^4$, we have $\overline{g_3}(x,y)=-cx^6$, we obtain that the origin of K is the only singular point of $g_3(x,y)$ over $(\mathbb{F}_q^\times)^2$.

By using equation (A.1.1), so we obtain that,

$$Z(s,g,\chi,\Delta_3)=\sum_{n=1}^{\infty}q^{-2n-6ns}\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\int_{(a,b)+(\mathfrak{p}O_K)^2}\chi(ac\ g_3(x,y))|g_3(x,y)|^s|dxdy|,$$
$$=\sum_{n=1}^{\infty}q^{-2n-6ns-2}\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\int_{O_K^2}\chi(ac\ g_3(a+\mathfrak{p}x,b+\mathfrak{p}y))|g_3(a+\mathfrak{p}x,b+\mathfrak{p}y)|^s|dxdy|.$$

Now we apply the change variables (A.1.3) to function g_3 and since that $\frac{\partial \overline{g_3}}{\partial x}(\overline{a},\overline{b})=-6c\overline{a}^5\neq 0$, we obtain

$$Z(s,g,\chi,\Delta_3)=$$
$$\sum_{n=1}^{\infty}q^{-2n-6ns-2}\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\int_{O_K^2}\chi(ac\ g_3(a+\mathfrak{p}x,b+\mathfrak{p}y))|g_3(a+\mathfrak{p}x,b+\mathfrak{p}y)|^s|dxdy|,$$
$$=\sum_{n=1}^{\infty}q^{-2n-6ns-2}\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\int_{O_K}\chi(ac\ (g_3(a,b)+\mathfrak{p}z_1))|g_3(a,b)+\mathfrak{p}z_1)|^s|dz_1|,$$
$$=\sum_{n=1}^{\infty}q^{-2n-6ns-2}I_{\Delta_3}(s,(a,b)),$$

where $I_{\Delta_3}(s,(a,b))=\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\int_{O_K}\chi(ac\,(g_3(a,b)+\mathfrak{p}z_1))|g_3(a,b)+\mathfrak{p}z_1)|^s|dz_1|$. Then given that $N=Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2:\overline{g}_3(\overline{a},\overline{b})=0\}=Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2:-\overline{ca}^6=0\}=0$, we have

$$I_{\Delta_3}(s,(a,b))=\begin{cases}(q-1)^2 & \textit{if}\ \ \chi=\chi_{triv}\\ \sum_{\substack{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}\\ \overline{g}_3(\overline{a},\overline{b})\neq 0}}\overline{\chi}(\overline{g}_3(\overline{a},\overline{b})) & \textit{if}\ \ \overline{\chi}=\chi_{triv}\\ 0 & \textit{all other cases.}\end{cases}$$

where $\overline{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}_q$. Then by applying similar arguments to the case above and (B.1.1),

and given that $Z(s,g,\chi,\Delta_3)=\sum_{n=1}^{\infty}q^{-2n-6ns-2}I_{\Delta_3}(s,(a,b))$, we have

$$Z(s,g,\chi,\Delta_3)=\begin{cases}\frac{(q-1)^2q^{-2-6s}}{1-q^{-2-6s}} & \text{if } \chi=\chi_{triv}\\ \chi(-\overline{c})\frac{q^{-2-6s}(1-q^{-1})^2}{(1-q^{-2-6s})} & \text{if } \chi^6=\chi_{triv}, \chi|_U=\chi_{triv}\\ 0 & \text{all other cases.}\end{cases}$$

(d) Case $Z(s,g,\chi,\Delta_4)$.

$$Z(s,g,\chi,\Delta_4)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int\limits_{\mathfrak{p}^{n+3m}O_K^{\times}\times\mathfrak{p}^{n+2m}O_K^{\times}}\chi(ac\ g(x,y))|g(x,y)|^s|dxdy|,$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{(-2-6s)n+(-5-18s)m}\int\limits_{O_K^{\times 2}}\mathcal{X}_1(g_4(x,y))\ |dxdy|.$$

where

$$\mathcal{X}_1(g_4(x,y))=\chi(ac\ g_4(x,y))|g_4(x,y)|^s,$$

and the polynomial $g_4(x,y)=(\mathfrak{p}^ny^3-x^2)^2(\mathfrak{p}^ny^3-cx^2)+\mathfrak{p}^{2n+2m}x^4y^4)$, with $\overline{g_4}(x,y)=-cx^6$, we obtain that the origin of K is the only singular point of $g_4(x,y)$ over $(\mathbb{F}_q^{\times})^2$.

By using equation (A.1.1), so we can assert that

$$Z(s,g,\chi,\Delta_4)=$$
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{(-2-6s)n+(-5-18s)m}\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\int\limits_{(a,b)+(\mathfrak{p}O_K)^2}\chi(ac\ g_4(x,y))|g_4(x,y)|^s|dxdy|,$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{(-2-6s)n+(-5-18s)m-2}\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\int_{O_K^2}\mathcal{X}_2(g_4(a+\mathfrak{p}x,b+\mathfrak{p}y))|dxdy|.$$

where $\mathcal{X}_2(g_4(a+\mathfrak{p}x,b+\mathfrak{p}y))=\chi(ac\ g_4(a+\mathfrak{p}x,b+\mathfrak{p}y))\ |g_4(a+\mathfrak{p}x,b+\mathfrak{p}y)|^s$.

Now we apply the change variables (A.1.3) to function g_4 and since that $\frac{\partial \overline{g_4}}{\partial x}(\overline{a},\overline{b}) = -6\overline{ca}^5 \neq 0$, we see that,

$$Z(s,g,\chi,\Delta_4) =$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{(-2-6s)n+(-5-18s)m-2} \sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}} \int_{O_K^2} \mathcal{X}_2((g_4(a+\mathfrak{p}x, b+\mathfrak{p}y)))\ |dxdy|,$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{(-2-6s)n+(-5-18s)m-2} I_{\Delta_4}(s,(a,b)),$$

where $I_{\Delta_4}(s,(a,b)) = \sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}} \int\limits_{O_K} \chi(ac\,(g_4(a,b)+\mathfrak{p}z_1))|g_4(a,b)+\mathfrak{p}z_1)|^s|dz_1|$, then we apply Lemma 1.2.2 and given that $N = Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2 : \overline{g}_4(\overline{a},\overline{b}) = 0\} = Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2 : -\overline{ca}^6 = 0\} = 0$, we get that

$$I_{\Delta_4}(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if}\ \ \chi = \chi_{triv} \\ \sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2 \\ \overline{g}_4(\overline{a},\overline{b})\neq 0}} \overline{\chi}(\overline{g}_4(\overline{a},\overline{b})) & \textit{if}\ \ \overline{\chi} = \chi_{triv} \\ 0 & \textit{all other cases,} \end{cases}$$

Finally, by applying (B.1.1) and since that

$$Z(s,g,\chi,\Delta_4) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{(-2-6s)n+(-5-18s)m-2} I_{\Delta_4}(s,(a,b)),$$

we conclude that

$$Z(s,g,\chi,\Delta_4) = \begin{cases} \frac{(1-q^{-1})^2 q^{-7-24s}}{(1-q^{-2-6s})(1-q^{-5-18s})} & \textit{if}\ \ \chi = \chi_{triv} \\ \chi(-\overline{c})\frac{q^{-7-24s}(1-q^{-1})^2}{(1-q^{-2-6s})(1-q^{-5-18s})} & \textit{if}\ \ \chi^6 = \chi_{triv}, \chi|_U = \chi_{triv} \\ 0 & \textit{all other cases,} \end{cases}$$

(e) Case $Z(s,g,\chi,\Delta_6)$.

$$Z(s,g,\chi,\Delta_6)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_{\mathfrak{p}^{3n+2m}O_K^{\times}\times\mathfrak{p}^{2n+m}O_K^{\times}}\chi(ac\ g(x,y))|g(x,y)|^s|dxdy|,$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{(-5-18s)n+(-3-9s)m}\int_{O_K^{\times 2}}\chi(ac(g_6(x,y)))|g_6(x,y)|^s\ |dxdy|,$$

where polynomial $g_6(x,y)=(y^3-\mathfrak{p}^m x^2)^2(y^3-c\mathfrak{p}^m x^2)+\mathfrak{p}^{2n+3m}x^4y^4$, we have $\overline{g_6}(x,y)=y^9$. Then we obtain that,

$$Z(s,g,\chi,\Delta_6)=$$
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{(-5-18s)n+(-3-9s)m}\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\int_{(a,b)+(\mathfrak{p}O_K)^2}\mathcal{X}(g_6(x,y))|dxdy|,$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{(-5-18s)n+(-3-9s)m-2}\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\int_{O_K^2}\mathcal{X}(g_6(a+\mathfrak{p}x,b+\mathfrak{p}y))\ |dxdy|,$$

where $\mathcal{X}(g_6(x,y))=\chi(ac(g_6(x,y)))|g_6(x,y)|^s$. Now we apply the change variables (A.1.3) to function g_6 and since that $\frac{\partial g_6}{\partial y}(\overline{a},\overline{b})=9(\overline{b}^8)\neq 0$, we obtain that,

$$Z(s,g,\chi,\Delta_6)=$$
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{(-5-18s)n+(-3-9s)m-2}\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\int_{O_K}\mathcal{X}(g_6((a,b)+\mathfrak{p}z_1))|\ |dz_1|,$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{(-5-18s)n+(-3-9s)m-2}I_{\Delta_6}(s,(a,b)),$$

where $I_{\Delta_6}(s,(a,b))=\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\int_{O_K}\mathcal{X}(g_6((a,b)+\mathfrak{p}z_1))|dz_1|$, then given that $N=Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2:\overline{g}_6(\overline{a},\overline{b})=0\}=Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2:\overline{b}^9=0\}=$

0, we obtain

$$I_{\Delta_6}(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if } \chi=\chi_{triv} \\ \sum_{\substack{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}\\ \overline{g}_6(\overline{a},\overline{b})\neq 0}} \overline{\chi}(\overline{g}_6(\overline{a},\overline{b})) & \textit{if } \overline{\chi}=\chi_{triv} \\ 0 & \textit{all other cases.} \end{cases}$$

Then,

$$I_{\Delta_6}(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if } \chi=\chi_{triv} \\ \sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}} \overline{\chi}(\overline{b}^9) & \textit{if } \overline{\chi}=\chi_{triv} \\ 0 & \textit{all other cases,} \end{cases}$$

where $\overline{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}_q$.

Now since that $\overline{\chi}^9=\chi_{triv}$ and $\chi|_U=\chi_{triv}$, $U=1+\mathfrak{p}O_K$ implies $\chi^9=\chi_{triv}$, we get

$$\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \overline{\chi}(b^9) = \begin{cases} (q-1)^2 & \textit{if } \chi^9=\chi_{triv},\chi|_U=\chi_{triv} \\ 0 & \textit{all other cases.} \end{cases} \tag{B.1.2}$$

Therefore,

$$I_{\Delta_6}(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if } \chi=\chi_{triv} \\ (q-1)^2 & \textit{if } \chi^9=\chi_{triv},\chi|_U=\chi_{triv} \\ 0 & \textit{all other cases.} \end{cases}$$

Finally, since that

$$Z(s,g,\chi,\Delta_6) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{(-5-18s)n+(-3-9s)m-2} I_{\Delta_6}(s,(a,b)),$$

we obtain

$$Z(s,g,\chi,\Delta_6)=\begin{cases}\frac{q^{-8-27s}(1-q^{-1})^2}{(1-q^{-3-9s})(1-q^{-5-18s})} & \textit{if } \chi=\chi_{triv}\\ \\ \frac{q^{-8-27s}(1-q^{-1})^2}{(1-q^{-3-9s})(1-q^{-5-18s})} & \textit{if } \chi^9=\chi_{triv}, \chi|_U=\chi_{triv}\\ \\ 0 & \textit{all other cases},\end{cases}$$

(f) Case $Z(s,g,\chi,\Delta_7)$.

$$\begin{aligned}Z(s,g,\chi,\Delta_7)&=\sum_{n=1}^{\infty}\int_{\mathfrak{p}^{2n}O_K^\times\times\mathfrak{p}^nO_K^\times}\chi(ac\ g(x,y))|g(x,y)|^s|dxdy|,\\ &=\sum_{n=1}^{\infty}q^{-3n-9ns}\int_{O_K^{\times 2}}\mathcal{X}(g_7(x,y))\ |dxdy|.\end{aligned}$$

where $\mathcal{X}(g_7(x,y))=\chi(ac(g_7(x,y)))$ and the polynomials

$$g_7(x,y)=(y^3-\mathfrak{p}^nx^2)^2(y^3-c\mathfrak{p}^nx^2)+\mathfrak{p}^{3n}x^4y^4,\ \text{with}\ \ \overline{g_7}(x,y)=y^9,$$

therefore the origin of K is the only singular point of $g_7(x,y)$ over $(\mathbb{F}_q^\times)^2$. Then we have,

$$\begin{aligned}&Z(s,g,\chi,\Delta_7)=\\ &\sum_{n=1}^{\infty}q^{-3n-9ns}\sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}}\int_{(a,b)+(\mathfrak{p}O_K)^2}\chi(ac\ g_7(x,y))|g_7(x,y)|^s|dxdy|\\ &=\sum_{n=1}^{\infty}q^{-3n-9ns-2}\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2}\int_{O_K^2}\mathcal{X}(g_7(a+\mathfrak{p}x,b+\mathfrak{p}y))\ |dxdy|.\end{aligned}$$

Now we apply the change variables (A.1.3) to function g_7 and since that $\frac{\partial\overline{g_7}}{\partial y}(\overline{a},\overline{b})=9\overline{b}^8\neq 0$, we obtain that,

$$
\begin{aligned}
&Z(s, g, \chi, \Delta_7) = \\
&\sum_{n=1}^{\infty} q^{-3n-9ns-2} \sum_{(\overline{a},\overline{b})\in\mathbb{F}_q^{\times 2}} \int_{O_K^2} \mathcal{X}(g_7(a+\mathfrak{p}x, b+\mathfrak{p}y))|dxdy| \\
&= \sum_{n=1}^{\infty} q^{-3n-9ns-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K^2} \mathcal{X}(g_7(a,b)+\mathfrak{p}z_1)\ |dz_1| \\
&= \sum_{n=1}^{\infty} q^{-3n-9ns-2} I_{\Delta_7}(s,(a,b)),
\end{aligned}
$$

where $I_{\Delta_7}(s,(a,b)) = \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K} \chi(ac\,(g_7(a,b)+\mathfrak{p}z_1))|g_7(a,b)+\mathfrak{p}z_1)|^s|dz_1|$, then since $N = Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2 : \overline{g}_7(\overline{a},\overline{b}) = 0\} = Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2 : b^9 = 0\} = 0$, we apply the argument above again, and for the equation (B.1.2) in

$$Z(s, g, \chi, \Delta_7) = \sum_{n=1}^{\infty} q^{-3n-9ns-2} I_{\Delta_7}(s,(a,b)),$$

we conclude

$$
Z(s, g, \chi, \Delta_7) = \begin{cases}
\frac{q^{-3-9s}(1-q^{-1})^2}{(1-q^{-3-9s})} & if\ \ \chi = \chi_{triv} \\
\\
\frac{q^{-3-9s}(1-q^{-1})^2}{(1-q^{-3-9s})} & if\ \ \chi^9 = \chi_{triv}, \chi|_U = \chi_{triv} \\
\\
0 & all\ other\ cases.
\end{cases}
$$

(g) Case $Z(s, g, \chi, \Delta_8)$.

$$Z(s,g,\chi,\Delta_8) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \int_{\mathfrak{p}^{2n+m}O_K^{\times}\times\mathfrak{p}^nO_K^{\times}} \chi(ac\ g(x,y))|g(x,y)|^s|dxdy|,$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{(-3-9s)n-m} \int_{O_K^{\times 2}} \chi(ac\ (g_8(x,y)))\ |g_8(x,y)|^s\ |dxdy|.$$

Since that polynomial $g_8(x,y) = (y^3-\mathfrak{p}^{n+2m}x^2)^2(y^3-c\mathfrak{p}^{n+2m}x^2)+\mathfrak{p}^{3n+4m}x^4y^4$ we have $\overline{g_8}(x,y) = y^9$, then we obtain that the origin of K is the only singular point of $g_8(x,y)$ over $(\mathbb{F}_q^{\times})^2$. By using equation (A.1.1), so we obtain that,

$$Z(s,g,\chi,\Delta_8) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{(-3-9s)n-m-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{(a,b)+(\mathfrak{p}O_K)^2} \chi(ac\ g_8(x,y))|g_8(x,y)|_K^s|dxdy|,$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{(-3-9s)n-m-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K^2} \mathcal{X}(g_8(a+\mathfrak{p}x,b+\mathfrak{p}y))|dxdy|,$$

where $\mathcal{X}(g_8(a+\mathfrak{p}x,b+\mathfrak{p}y))) = \chi(ac\ g_8(a+\mathfrak{p}x,b+\mathfrak{p}y))|g_8(a+\mathfrak{p}x,b+\mathfrak{p}y)|^s$. Now we apply the change variables (A.1.3) to function g_8 and since that $\frac{\partial\overline{g_8}}{\partial y}(\overline{a},\overline{b}) = 9\overline{b}^8 \neq 0$, we can assert that,

$$Z(s,g,\chi,\Delta_8) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{(-3-9s)n-m-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K} \chi(ac\ (g_8(a,b)+\mathfrak{p}z_1))|g_8(a,b)+\mathfrak{p}z_1)|^s|dz_1|,$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{(-3-9s)n-m-2} I_{\Delta_8}(s,(a,b)),$$

where $I_{\Delta_8}(s,(a,b))=\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2}\int_{O_K}\chi(ac\,(g_8(a,b)+\mathfrak{p}z_1))|g_8(a,b)+\mathfrak{p}z_1)|^s|dz_1|$, and since $N=Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2:\overline{g}_8(\overline{a},\overline{b})=0\}=Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2:\overline{b}^9=0\}=0$ we yields,

$$Z(s,g,\chi,\Delta_8)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{(-3-9s)n-m-2}I_{\Delta_8}(s,(a,b),$$

and by applying (B.1.2) we conclude that

$$Z(s,g,\chi,\Delta_8=\begin{cases}\frac{q^{-4-9s}(1-q^{-1})}{(1-q^{-3-9s})} & if\ \ \chi=\chi_{triv}\\ \frac{q^{-4-9s}(1-q^{-1})}{(1-q^{-3-9s})} & if\ \ \chi^9=\chi_{triv},\chi|_U=\chi_{triv}\\ 0 & all\ other\ cases.\end{cases}$$

(h) Case $Z(s,g,\chi,\Delta_9)$.

$$Z(s,g,\chi,\Delta_9)=\sum_{n=1}^{\infty}\int_{\mathfrak{p}^nO_K^{\times}\times O_K^{\times}}\chi(ac\ g(x,y))|g(x,y)|^s|dxdy|,$$
$$=\sum_{n=1}^{\infty}q^{-n}\int_{O_K^{\times 2}}\chi(ac\ (g_9(x,y)))\ |g_9(x,y)|^s\ |dxdy|.$$

Since that polynomial $g_9(x,y)=(y^3-\mathfrak{p}^{2n}x^2)^2(y^3-c\mathfrak{p}^{2n}x^2)+\mathfrak{p}^{4n}x^4y^4$, we have $\overline{g}_9(x,y)=y^9$ then we obtain that the origin of K is the only singular point of $g_9(x,y)$ over $(\mathbb{F}_q^{\times})^2$.

Then we obtain that,

$$Z(s,g,\chi,\Delta_9)=$$
$$\sum_{n=1}^{\infty} q^{-n} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{(a,b)+(\mathfrak{p}O_K)^2} \chi(ac\ g_9(x,y))|g_9(x,y)|^s|dxdy|$$
$$=\sum_{n=1}^{\infty} q^{-n-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K^2} \chi(ac\ g_9(a+\mathfrak{p}x,b+\mathfrak{p}y))|g_9(a+\mathfrak{p}x,b+\mathfrak{p}y)|^s|dxdy|.$$

Now we apply the change variables (A.1.3) to function g_9 and since that $\frac{\partial \overline{g_9}}{\partial y}(\overline{a},\overline{b})=9\overline{b}^8\neq 0$, we get,

$$Z(s,g,\chi,\Delta_9)=$$
$$\sum_{n=1}^{\infty} q^{-n-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{(O_K^{\times})^2} \chi(ac\ g_9(a+\mathfrak{p}x,b+\mathfrak{p}y))|g_9(a+\mathfrak{p}x,b+\mathfrak{p}y)|^s|dxdy|$$
$$=\sum_{n=1}^{\infty} q^{-n-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K} \chi(ac\ (g_9(a,b)+\mathfrak{p}z_1))|g_9(a,b)+\mathfrak{p}z_1)|^s|dz_1|$$
$$=\sum_{n=1}^{\infty} q^{-n-2} I_{\Delta_9}(s,(a,b)),$$

where $I_{\Delta_9}(s,(a,b))=\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K} \chi(ac\ (g_9(a,b)+\mathfrak{p}z_1))|g_9(a,b)+\mathfrak{p}z_1)|^s|dz_1|$, then given that

$$N=Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2:\overline{g}_9(\overline{a},\overline{b})=0\}=Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2:\overline{b}^9=0\}=0$$

we obtain,

$$I_{\Delta_9}(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if } \chi=\chi_{triv} \\ \sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2 \\ \overline{g}_9(\overline{a},\overline{b})\neq 0}} \overline{\chi}(\overline{g}_9(\overline{a},\overline{b})) & \textit{if } \overline{\chi}=\chi_{triv} \\ 0 & \textit{all other cases,} \end{cases}$$

where $\overline{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}$.

Now since that $Z(s,g,\chi,\Delta_9)=\sum_{n=1}^{\infty} q^{-n-2} I_{\Delta_9}(s,(a,b))$, then as in the case $Z(s,g,\chi,\Delta_6)$, the equation $(B.1.2)$ gives

$$Z(s,g,\chi,\Delta_9) = \begin{cases} q^{-1}(1-q^{-1}) & \textit{if } \chi=\chi_{triv} \\ q^{-1}(1-q^{-1}) & \textit{if } \chi^9=\chi_{triv}, \chi|_U=\chi_{triv} \\ 0 & \textit{all other cases.} \end{cases}$$

Now we are going to find $Z(s,f,\chi,\Delta_i)$ for $i=1,2,3,4,6,7,8,9$ with the computes above:

When $\chi=\chi_{triv}$.

$$Z(s,f,\chi_{triv}) = 2q^{-1}(1-q^{-1}) + \frac{q^{-2-6s}(1-q^{-1})}{(1-q^{-2-6s})} + \frac{q^{-7-24s}(1-q^{-1})^2}{(1-q^{-2-6s})(1-q^{-5-18s})} + \frac{q^{-8-27s}(1-q^{-1})^2}{(1-q^{-3-9s})(1-q^{-5-18s})} + \frac{q^{-3-9s}(1-q^{-1})}{(1-q^{-3-9s})}$$

When $\chi\neq\chi_{triv}$ and $\chi_{|}1+\mathfrak{p}O_K=\chi_{triv}$ we have several cases: if $\chi^6=\chi_{triv}$, we have

$$Z(s,f,\chi) = \chi(-\overline{c})\left(q^{-1}(1-q^{-1}) + \frac{q^{-3-6s}(1-q^{-1})}{(1-q^{-2-6s})} + \frac{q^{-2-6s}(1-q^{-1})^2}{(1-q^{-2-6s})}\right) + \chi(-\overline{c})\left(\frac{q^{-7-24s}(1-q^{-1})^2}{(1-q^{-2-6s})(1-q^{-5-18s})}\right).$$

In the case where $\chi^9=\chi_{triv}$, we obtain

$$Z(s,f,\chi)=\frac{q^{-8-27s}(1-q^{-1})^2}{(1-q^{-3-9s})(1-q^{-5-18s})}+\frac{q^{-3-9s}(1-q^{-1})^2}{(1-q^{-3-9s})}$$
$$+\frac{q^{-4-9s}(1-q^{-1})}{(1-q^{-3-9s})}+q^{-1}(1-q^{-1}).$$

In all other cases, $Z(s,f,\chi)=0$.

B.2 Computation of $Z(s,g,\chi,\Delta_5)$

(An integral on a degenerate face in the sense Kouchnirenko).

$$Z(s,g,\chi,\Delta_5)=\sum_{n=1}^{\infty}\int_{\mathfrak{p}^{3n}O_K^\times\times\mathfrak{p}^{2n}O_K^\times}\chi(ac\ g(x,y))|g(x,y)|^s|dxdy|,$$
$$=\sum_{n=1}^{\infty}q^{-5n-18ns}\int_{O_K^{\times 2}}\chi(ac((y^3-x^2)^2(y^3-cx^2)+\mathfrak{p}^{2n}x^4y^4)$$
$$|(y^3-x^2)^2(y^3-cx^2)+\mathfrak{p}^{2n}x^4y^4|^s|dxdy|.$$

Let $g^{(n)}(x,y)=(y^3-x^2)^2+\mathfrak{p}^{2n}x^4y^4$, for $n\geqslant 1$. For compute the integral,

$I(s,g^{(n)},\chi)=\int_{O_K^{\times 2}}\chi(ac((y^3-x^2)^2(y^3-cx^2)+\mathfrak{p}^{2n}x^4y^4))|(y^3-x^2)^2(y^3-cx^2)+\mathfrak{p}^{2n}x^4y^4|^s|dxdy|$, for $n\geqslant 1$, we use the following change of variables:

$$\Phi:\begin{array}{ll}O_K^{\times 2} & \to O_K^{\times 2}\\ (x,y) & \longmapsto (x^3y,x^2y)\end{array}$$

The map Φ gives an analytic bijection of $O_K^{\times 2}$ onto itself and preserves the Haar measure since that its Jacobian $J_\Phi(x,y)=x^4y$ satisfies $|J_\Phi(x,y)|_K=1$, for every $x,y\in O_K^\times$. Thus

$g^{(n)}\circ\Phi(x,y)=x^{18}y^6\widetilde{g^{(n)}}(x,y)$, with

$$\widetilde{g^{(n)}}(x,y)=(y-1)^2(y-c)+\mathfrak{p}^{2n}x^2y^2, \tag{B.2.1}$$

Then we have that,

$$I(s,g^{(n)},\chi)=\int_{O_K^{\times 2}}\chi(ac(x^{18}y^6\widetilde{g^{(n)}}(x,y)))|\widetilde{g^{(n)}}(x,y)|^s|dxdy|.$$

In order to compute the integral $I(s,g^{(n)},\chi), n\geqslant 1$, we decompose $O_K^{\times 2}$ as follows:

$$O_K^{\times 2}=\left(O_K^\times\times\{y_0+\mathfrak{p}O_K|y_0\not\equiv 1,c(mod\ \mathfrak{p})\}\right)\cup\left(O_K^\times\times\{1+\mathfrak{p}O_K\}\right)\cup\left(O_K^\times\times\{c+\mathfrak{p}O_K\}\right), \tag{B.2.2}$$

where y_0 runs through a set of representatives of $\mathbb{F}_q^\times$ in O_K. From partition (B.2.1) and formula (B.2), it follows that,

$$\begin{aligned}I(s,g^{(n)},\chi)=&\int_{O_K^\times\times\{y_0+\mathfrak{p}O_K\}}\chi(ac(x^{18}y^6\widetilde{g^{(n)}}(x,y)))|\widetilde{g^{(n)}}(x,y)|^s|dxdy|\\&+\int_{O_K^\times\times\{1+\mathfrak{p}O_K\}}\chi(ac(x^{18}y^6\widetilde{g^{(n)}}(x,y)))|\widetilde{g^{(n)}}(x,y)|^s|dxdy|\\&+\int_{O_K^\times\times\{c+\mathfrak{p}O_K\}}\chi(ac(x^{18}y^6\widetilde{g^{(n)}}(x,y)))|\widetilde{g^{(n)}}(x,y)|^s|dxdy|.\end{aligned}$$

The integral I admits the following expansion:

$$I(s,g^{(n)},\chi)=$$
$$q^{-1}\sum_{y_0\not\equiv 1,c(mod\mathfrak{p})}\int_{O_K^\times\times O_K}\chi(ac(x^{18}(y_0+\mathfrak{p}y)^6\widetilde{g^{(n)}}(x,y_0+\mathfrak{p}y)))|\widetilde{g^{(n)}}(x,y_0+\mathfrak{p}y)|^s|dxdy|$$
$$+q^{-1}\int_{O_K^\times\times O_K}\chi(ac(x^{18}(1+\mathfrak{p}y)^6\widetilde{g^{(n)}}(x,1+\mathfrak{p}y)))|\widetilde{g^{(n)}}(x,1+\mathfrak{p}y)|^s|dxdy|$$
$$+q^{-1}\int_{O_K^\times\times O_K}\chi(ac(x^{18}(c+\mathfrak{p}y)^6\widetilde{g^{(n)}}(x,c+\mathfrak{p}y)))|\widetilde{g^{(n)}}(x,c+\mathfrak{p}y)|^s|dxdy|.$$

Now we use $O_K=\bigsqcup_{j=0}^{\infty}\mathfrak{p}^jO_k^\times$ and it follows that,

$$I(s,g^{(n)},\chi)=$$
$$\sum_{y_0\not\equiv 1,c(mod\mathfrak{p})}\sum_{j=0}^{\infty}q^{-1-j}\int\limits_{O_K^{\times 2}}\mathcal{X}(\widetilde{g^{(n)}}(x,y_0+\mathfrak{p}^{j+1}y))\ |dxdy|$$
$$+\sum_{j=0}^{\infty}q^{-1-j}\int\limits_{O_K^{\times 2}}\mathcal{X}(\widetilde{g^{(n)}}(x,1+\mathfrak{p}^{j+1}y))\ |dxdy|$$
$$+\sum_{j=0}^{\infty}q^{-1-j}\int\limits_{O_K^{\times}\times O_K}\mathcal{X}(\widetilde{g^{(n)}}(x,c+\mathfrak{p}^{j+1}y)))\ |dxdy|.$$

where

$$\mathcal{X}(\widetilde{g^{(n)}}(x,y_0+\mathfrak{p}^{j+1}y))=\chi(ac(x^{18}(y_0+\mathfrak{p}^{j+1}y)^6\widetilde{g^{(n)}}(x,y_0+\mathfrak{p}^{j+1}y)))|\widetilde{g^{(n)}}(x,y_0+\mathfrak{p}^{j+1}y)|^s$$
$$\mathcal{X}(\widetilde{g^{(n)}}(x,1+\mathfrak{p}^{j+1}y))=\chi(ac(x^{18}(1+\mathfrak{p}^{j+1}y)^6\widetilde{g^{(n)}}(x,1+\mathfrak{p}^{j+1}y)))|\widetilde{g^{(n)}}(x,1+\mathfrak{p}^{j+1}y)|^s$$
$$\mathcal{X}(\widetilde{g^{(n)}}(x,c+\mathfrak{p}^{j+1}y)))=\chi[ac(x^{18}(c+\mathfrak{p}^{j+1}y)^6\widetilde{g^{(n)}}(x,c+\mathfrak{p}^{j+1}y)))|\widetilde{g^{(n)}}(x,c+\mathfrak{p}^{j+1}y)|^s$$

Then we can write, $I(s,g^{(n)},\chi)=J_1(s,g^{(n)},\chi)+J_2(s,g^{(n)},\chi)+J_3(s,g^{(n)},\chi)$,

where

$$J_1(s, g^{(n)}, \chi) = \sum_{y_0 \not\equiv 1, c (mod \mathfrak{p})} \sum_{j=0}^{\infty} q^{-1-j} \int_{O_K^{\times 2}} \mathcal{X}(\widetilde{g^{(n)}}(x, y_0 + \mathfrak{p}^{j+1} y)) \ |dxdy|$$

$$J_2(s, g^{(n)}, \chi) = \sum_{j=0}^{\infty} q^{-1-j} \int_{O_K^{\times 2}} \mathcal{X}(\widetilde{g^{(n)}}(x, 1 + \mathfrak{p}^{j+1} y)) \ |dxdy|$$

$$J_3(s, g^{(n)}, \chi) = \sum_{j=0}^{\infty} q^{-1-j} \int_{O_K^{\times} \times O_K} \mathcal{X}(\widetilde{g^{(n)}}(x, c + \mathfrak{p}^{j+1} y))) \ |dxdy|$$

Then we can expand $J_2(s, g^{(n)}, \chi)$ and $J_3(s, g^{(n)}, \chi)$ as following

$$J_2(s, g^{(n)}, \chi) = \sum_{j=0}^{n-2} q^{-1-j-(2+2j)s} \int_{O_K^{\times 2}} \chi(ac \ g_2(x, y))|dxdy|$$

$$+q^{-n-2ns} \int_{O_K^{\times 2}} \chi(ac \ g_3(x, y))|g_3(x, y)|^s|dxdy|$$

$$+ \sum_{j=n}^{\infty} q^{-1-j-2ns} \int_{O_K^{\times 2}} \chi(ac \ g_4(x, y))|dxdy|.$$

$$J_3(s, g^{(n)}, \chi) = \sum_{j=0}^{2n-2} q^{-1-j-(j+1)s} \int_{O_K^{\times 2}} \chi(ac \ g_5(x, y))|dxdy|$$

$$+q^{-2ns-2n} \int_{O_K^{\times 2}} \chi(ac \ g_6(x, y))|g_6(x, y)|^s|dxdy|$$

$$+ \sum_{j=2n}^{\infty} q^{-1-j-2ns} \int_{O_K^{\times 2}} \chi(ac \ g_7(x, y))|dxdy|,$$

So, we can write

$$J_1(s,g^{(n)},\chi)=I_1(s,g^{(n)},\chi)$$
$$J_2(s,g^{(n)},\chi)=I_2(s,g^{(n)},\chi)+I_3(s,g^{(n)},\chi)+I_4(s,g^{(n)},\chi)$$
$$J_3(s,g^{(n)},\chi)=I_5(s,g^{(n)},\chi)+I_6(s,g^{(n)},\chi)+I_7(s,g^{(n)},\chi)$$

where,

$$I_1=\sum_{y_0\not\equiv 1,c(mod\mathfrak{p})}\sum_{j=0}^{\infty}q^{-1-j}\int_{O_K^{\times 2}}\chi(ac\ g_1(x,y))|dxdy|,$$
$$I_2=\sum_{j=0}^{n-2}q^{-1-j-(2+2j)s}\int_{O_K^{\times 2}}\chi(ac\ g_2(x,y))|dxdy|,$$
$$I_3=q^{-n-2ns}\int_{O_K^{\times 2}}\chi(ac\ g_3(x,y))|g_3(x,y)|^s|dxdy|,$$
$$I_4=\sum_{j=n}^{\infty}q^{-1-j-2ns}\int_{O_K^{\times 2}}\chi(ac\ g_4(x,y))|dxdy|,$$
$$I_5=\sum_{j=0}^{2n-2}q^{-1-j-(j+1)s}\int_{O_K^{\times 2}}\chi(ac\ g_5(x,y))|dxdy|,$$
$$I_6=q^{-2ns-2n}\int_{O_K^{\times 2}}\chi(ac\ g_6(x,y))|g_6(x,y)|^s|dxdy,|$$
$$I_7=\sum_{j=2n}^{\infty}q^{-1-j-2ns}\int_{O_K^{\times 2}}\chi(ac\ g_7(x,y))|dxdy|,$$

and,

$$\begin{aligned}
g_1(x,y) =& x^{18}(y_0+\mathfrak{p}^{j+1}y)^6((y_0-1+\mathfrak{p}^{j+1}y)^2(y_0+\mathfrak{p}^{j+1}y-c)+\mathfrak{p}^{2n}x^2(y_0+\mathfrak{p}^{j+1}y)^2),\\
g_2(x,y) =& [x^{18}(1+\mathfrak{p}^{j+1}y)^6][y^2(1-c+\mathfrak{p}^{j+1}y)+\mathfrak{p}^{2n-(2+2j)}x^2(1+\mathfrak{p}^{j+1}y)^2)],\\
g_3(x,y) =& x^{18}(1+\mathfrak{p}^{n}y)^6[(y^2(1-c+\mathfrak{p}^{n})+x^2(1+\mathfrak{p}^{n}y)^2],\\
g_4(x,y) =& x^{18}(1+\mathfrak{p}^{j+1}y)^6[(\mathfrak{p}^{2+2j-2n}y^2(1-c+\mathfrak{p}^{j+1}y)+x^2(1+\mathfrak{p}^{j+1}y)^2],\\
g_5(x,y) =& [x^{18}(c+\mathfrak{p}^{j+1}y)^6][y(c-1+\mathfrak{p}^{j+1}y)^2+\mathfrak{p}^{2n-(1+j)}x^2(c+\mathfrak{p}^{j+1}y)^2)],\\
g_6(x,y) =& x^{18}(c+\mathfrak{p}^{2n}y)^6[(y(c-1+\mathfrak{p}^{2n}y)^2+x^2(c+\mathfrak{p}^{2n}y)^2],\\
g_7(x,y) =& x^{18}(c+\mathfrak{p}^{j+1}y)^6[(\mathfrak{p}^{1+j-2n}y(c-1+\mathfrak{p}^{j+1}y)^2+x^2(c+\mathfrak{p}^{j+1}y)^2],
\end{aligned}$$

where the reduction of the coefficients of each function is

$$\begin{aligned}
\overline{g_1}(x,y) =& x^{18}(y_0)^7(y_0-1)^2 & \overline{g_2}(x,y) =& x^{18}y^2(1-c),\\
\overline{g_3}(x,y) =& x^{18}y^2(1-c)+x^{20} & \overline{g_4}(x,y) =& x^{20},\\
\overline{g_5}(x,y) =& x^{18}y^2c^6(c-1) & \overline{g_6}(x,y) =& x^{18}yc^6(c-1)^2+x^2c^2,\\
\overline{g_7}(x,y) =& x^{20}c^8.
\end{aligned}$$

Note that we can find every integral $I_i, i=1,2,3,4$ and we compute $Z(s,g,\chi,\Delta_5) = \sum_{n=1}^{\infty} q^{-5n-18ns} I(s,g^{(n)},\chi)$, where $I(s,g^{(n)},\chi) = I_1+I_2+I_3+I_4+I_5+I_6+I_7$

Now we'll find every integral I_i for $i=1,2,3,4,5,6,7$.

(a) $I_1 = \sum_{y_0 \not\equiv 1,c (mod \mathfrak{p})} \sum_{j=0}^{\infty} q^{-1-j} \int\limits_{O_K^{\times 2}} \chi(ac\, g_1(x,y))|dxdy|$.

Since that the polynomial $g_1(x,y) = x^{18}(y_0+\mathfrak{p}^{j+1}y)^6((y_0-1+\mathfrak{p}^{j+1}y)^2(y_0+\mathfrak{p}^{j+1}y-c)+\mathfrak{p}^{2n}x^2(y_0+\mathfrak{p}^{j+1}y)^2)$, we have $\overline{g_1}(x,y) = x^{18}y_0^7(y_0-1)^2$.

By using equation (A.1.1), so we obtain that,

$$I_1 = \sum_{y_0 \not\equiv 1, c (mod \mathfrak{p})} \sum_{j=0}^{\infty} q^{-1-j} \sum_{(\overline{a},\overline{b}) \in (\mathbb{F}_q^\times)^2} \int_{(a,b)+(\mathfrak{p}O_K)^2} \chi(ac\ g_1(x,y))|dxdy|$$

$$= \sum_{y_0 \not\equiv 1, c (mod \mathfrak{p})} \sum_{j=0}^{\infty} q^{-3-j} \sum_{(\overline{a},\overline{b}) \in (\mathbb{F}_q^\times)^2} \int_{O_K^2} \chi(ac\ g_1(a+\mathfrak{p}x, b+\mathfrak{p}y))|dxdy|.$$

Now we apply the change variables (A.1.3) to function g_1 and since that $\frac{\partial \overline{g_1}}{\partial x}(\overline{a},\overline{b}) = 18y_0^7(y_0-1)^2\overline{a}^{17} \neq 0$, we obtain that,

$$I_1 = \sum_{y_0 \not\equiv 1, c (mod \mathfrak{p})} \sum_{j=0}^{\infty} q^{-3-j} \sum_{(\overline{a},\overline{b}) \in \mathbb{F}_q^{\times 2}} \int_{O_K} \chi(ac\ (g_1(a,b)+\mathfrak{p}z_1))|dz_1|$$

$$= \sum_{y_0 \not\equiv 1, c (mod \mathfrak{p})} \sum_{j=0}^{\infty} q^{-3-j} \overline{I_1}(s,(a,b)),$$

where $\overline{I_1}(s,(a,b)) = \sum_{(\overline{a},\overline{b}) \in (\mathbb{F}^\times)^2} \int_{O_K} \chi(ac\ (g_1(a,b)+\mathfrak{p}z_1))|dz_1|$, then we apply the Lemma 1.2.2, and given that $N = Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^\times)^2 : \overline{g}_1(\overline{a},\overline{b}) = 0\} = Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^\times)^2 : \overline{a}^{18}y_0^7(y_0-1)^2 = 0\} = 0$, we obtain

$$\overline{I_1}(s,(a,b)) = \begin{cases} (q-1)^2 & \text{if } \chi = \chi_{triv} \\ \sum_{\substack{(\overline{a},\overline{b}) \in (\mathbb{F}_q^\times)^2 \\ \overline{g}_1(\overline{a},\overline{b}) \neq 0}} \overline{\chi}(\overline{g}_1(\overline{a},\overline{b})) & \text{if } \overline{\chi} = \chi_{triv} \\ 0 & \text{all other cases,} \end{cases}$$

where $\overline{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}$. Then,

$$\overline{I_1}(s,(a,b)) = \begin{cases} (q-1)^2 & \text{if } \chi = \chi_{triv} \\ \sum_{\substack{(\overline{a},\overline{b}) \in (\mathbb{F}_q^\times)^2 \\ (\overline{a}^{18}\overline{y_0}^7(\overline{y_0}-1)^2) \neq 0}} \overline{\chi}(\overline{a}^{18}\overline{y_0}^7(\overline{y_0}-1)^2) & \text{if } \overline{\chi} = \chi_{triv} \\ 0 & \text{all other cases,} \end{cases}$$

Therefore,

$$\overline{I_1}(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if } \ \chi = \chi_{triv} \\ \sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2 \\ (\overline{a}^{18}\overline{y_0}^7(\overline{y_0}-1)^2)\neq 0}} \overline{\chi}^{18}(\overline{a})\overline{\chi}(\overline{y_0}^7(\overline{y_0}-1)^2) & \textit{if } \ \overline{\chi} = \chi_{triv} \\ 0 & \textit{all other cases,} \end{cases}$$

where $U = 1+\mathfrak{p}O_K$. Now since that $\overline{\chi}^{18} = \chi_{triv}$ and $\chi|_U = \chi_{triv}$ implies $\chi^{18} = \chi_{triv}$, we get

$$\overline{I_1}(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if } \ \chi = \chi_{triv} \\ \chi^7(\overline{y_0})\chi^2(\overline{y_0}-1)(q-1)^2 & \textit{if } \chi^{18} = \chi_{triv}, \chi|_U = 1 \\ 0 & \textit{all other cases.} \end{cases}$$

Finally, since that $I_1 = \sum_{y_0\not\equiv 1,c(mod\mathfrak{p})}\sum_{j=0}^{\infty} q^{-3-j}\overline{I_1}(s,(a,b))$, we obtain

$$I_1 = \begin{cases} q^{-1}(q-3)(1-q^{-1}) & \textit{if } \ \chi = \chi_{triv} \\ \chi^7(\overline{y_0})\chi^2(\overline{y_0}-1)q^{-1}(q-3)(1-q^{-1}) & \textit{if } \ \chi^{18} = \chi_{triv}, \chi|_U = \chi_{triv} \\ 0 & \textit{all other cases.} \end{cases}$$

(b) $I_2 = \sum_{j=0}^{n-2} q^{-1-j-(2+2j)s} \int\limits_{O_K^{\times 2}} \chi(ac\ g_2(x,y))|dxdy|$. Since that polynomial

$$g_2(x,y) = [x^{18}(1+\mathfrak{p}^{j+1}y)^6][y^2(1-c+\mathfrak{p}^{j+1}y)+\mathfrak{p}^{2n-(2+2j)}x^2(1+\mathfrak{p}^{j+1}y)^2)],$$

we have $\overline{g_2}(x,y)=x^{18}y^2(1-c)$. Then we get,

$$I_2=\sum_{j=0}^{n-2}q^{-1-j-(2+2j)s}\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2}\int_{(a,b)+(\mathfrak{p}O_K)^2}\chi(ac\ g_2(x,y))|dxdy|$$
$$=\sum_{j=0}^{n-2}q^{-3-j-(2+2j)s}\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2}\int_{O_K^2}\chi(ac\ g_2(a+\mathfrak{p}x,b+\mathfrak{p}y))|dxdy|.$$

Now we apply the change variables (A.1.3) to function g_2 and since that $\frac{\partial\overline{g_2}}{\partial x}(\overline{a},\overline{b})=18(\overline{a}^{17})(\overline{b}^2)(1-c)\neq 0$, we obtain,

$$I_2=\sum_{j=0}^{n-2}q^{-3-j-(2+2j)s}\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2}\int_{O_K}\chi(ac\ (g_2(a,b)+\mathfrak{p}z_1))|dz_1|$$
$$=\sum_{j=0}^{n-2}q^{-3-j-(2+2j)s}\overline{I}_2(s,(a,b)),$$

where $\overline{I}_2(s,(a,b))=\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2}\int_{O_K}\chi(ac\ (g_2(a,b)+\mathfrak{p}z_1))|dz_1|$.

Now given $N=Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2:\overline{g}_2(\overline{a},\overline{b})=0\}=Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2:\overline{a}^{18}\overline{b}^2(1-\overline{c})=0\}=0$, we can assert that

$$\overline{I}_2=\begin{cases}(q-1)^2 & \textit{if}\ \ \chi=\chi_{triv}\\ \overline{\chi}(1-\overline{c})(q-1)^2 & \textit{if}\ \ \overline{\chi}^2=\chi_{triv}\\ 0 & \textit{all other cases,}\end{cases}.$$

Given that $\overline{\chi}^2=\chi_{triv}$ and $\chi|_U=\chi_{triv}$ implies $\chi^2=\chi_{triv}$, we get

$$\overline{I}_2=\begin{cases}(q-1)^2 & \textit{if}\ \ \chi=\chi_{triv}\\ \overline{\chi}(1-\overline{c})(q-1)^2 & \textit{if}\ \ \chi^2=\chi_{triv},\chi|_U=\chi_{triv}\\ 0 & \textit{all other cases,}\end{cases}$$

where $U = 1+\mathfrak{p}O_K$.

Finally, since that $I_2 = \sum_{j=0}^{n-2} q^{-3-j-(2+2j)s}\overline{I}_2(s,(a,b))$, we conclude that

$$I_2 = \begin{cases} \frac{q^{-1-2s}(1-q^{(n-1)(-1-2s)})(1-q^{-1})^2}{1-q^{-1-2s}} & \textit{if } \chi = \chi_{triv} \\ \\ \overline{\chi}(1-\overline{c})\frac{q^{-1-2s}(1-q^{(n-1)(-1-2s)})(1-q^{-1})^2}{1-q^{-1-2s}}, & \textit{if } \chi^2 = \chi_{triv}, \chi|_U = \chi_{triv} \\ \\ 0, & \textit{all other cases.} \end{cases}$$

(c) $I_3 = q^{-n-2ns} \int\limits_{O_K^{\times 2}} \chi(ac\ g_3(x,y))|g_3(x,y)|^s\ |dxdy|$.

Since that polynomial $g_3(x,y) = x^{18}(1+\mathfrak{p}^n y)^6[(y^2(1-c+\mathfrak{p}^n y)+x^2(1+\mathfrak{p}^n y)^2]$, we have $\overline{g_3}(x,y) = x^{18}y^2(1-c)+x^{20}$. Then we get that,

$$I_3 = q^{-n-2ns} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int\limits_{(a,b)+(\mathfrak{p}O_K)^2} \chi(ac\ g_3(x,y))|g_3(x,y)|^s|dxdy|$$

$$= q^{-n-2ns-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int\limits_{O_K^2} \chi(ac\ g_3(a+\mathfrak{p}x,b+\mathfrak{p}y))|g_3(a+\mathfrak{p}x,b+\mathfrak{p}y)|^s|dxdy|.$$

Now we apply the change variables (A.1.3) to function g_3, and like $\frac{\partial\overline{g_3}}{\partial y}(\overline{a},\overline{b}) = 2(\overline{a}^{18})(\overline{b}) \neq 0$, we obtain

$$I_3 = q^{-n-2ns-2} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int\limits_{O_K} \chi(ac\ (g_3(a,b)+\mathfrak{p}z_1))|g_3(a,b)+\mathfrak{p}z_1|^s|dz_1|$$

$$= q^{-n-2ns-2}\overline{I}_3(s,(a,b)),$$

where $\overline{I}_3(s,(a,b)) = \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int\limits_{O_K} \chi(ac\ (g_3(a,b)+\mathfrak{p}z_1))|g_3(a,b)+\mathfrak{p}z_1|^s|dz_1|$.

Now given that

$$\begin{aligned} N_1 =& Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^\times)^2 : \overline{g}_3(\overline{a},\overline{b}) = 0\} \\ =& Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^\times)^2 : \overline{a}^{18}(\overline{b}^2(1-\overline{c})+\overline{a}^2) = 0\}, \end{aligned}$$

$$\overline{I}_{3,1}(s,(a,b)) = \begin{cases} \frac{q^{-s}(1-q^{-1})N_1}{(1-q^{-1-s})} & \textit{if } \chi = \chi_{triv} \\ \\ 0 & \textit{all other cases.} \end{cases}$$

and

$$\overline{I}_{3,2}(s,(a,b)) = \begin{cases} (q-1)^2 - N_1 & \textit{if } \chi = \chi_{triv} \\ \\ \sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2 \\ \overline{g}_3(\overline{a},\overline{b})\neq 0}} \chi(ac(\overline{g}_3(\overline{a},\overline{b})) & \textit{if } \chi|_U = \chi_{triv} \\ \\ 0 & \textit{in other case,} \end{cases}$$

where $U = 1 + \mathfrak{p}O_K$.

Since that $\overline{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}_q$, we have that

$$\overline{I}_{3,2}(s,(a,b)) = \begin{cases} (q-1)^2 - N_1 & \textit{if } \chi = \chi_{triv} \\ \\ \sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2 \\ \overline{a}^{18}(\overline{b}^2(1-\overline{c})+\overline{a}^2)\neq 0}} \overline{\chi}(\overline{a}^{18}(\overline{b}^2(1-\overline{c})+\overline{a}^2)) & \textit{if } \overline{\chi} = \chi_{triv} \\ \\ 0 & \textit{all other cases.} \end{cases}$$

Given that $\overline{\chi} = \chi_{triv}$ and $\chi|_U = \chi_{triv}$ implies $\chi = \chi_{triv}$, we get

$$\overline{I}_{3,2}(s,(a,b)) = \begin{cases} (q-1)^2 - N_1 + T_2 & \textit{if } \chi = \chi_{triv} \\ \\ 0 & \textit{all other cases.} \end{cases}$$

By writing,

$$T_2 = \sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2 \\ (a^{18}(b^2(1-\overline{c})+a^2)\neq 0}} \chi(a^{18}(b^2(1-\overline{c})+a^2)),$$

Finally, since that $I_3 = q^{-n-2ns-2}\left(\overline{I}_{3,1}(s,(a,b)) + \overline{I}_{3,2}(s,(a,b))\right)$, we obtain that

$$I_3 = \begin{cases} q^{-n-2ns-2}\left(\frac{q^{-s}(1-q^{-1})N_1}{(1-q^{-1-s})} + (q-1)^2 - N_1 + T_2\right) & \textit{if } \chi = \chi_{triv} \\ \\ 0 & \textit{all other cases.} \end{cases}$$

(d) $I_4 = \sum_{j=n}^{\infty} q^{-1-j-2ns} \int\limits_{O_K^{\times 2}} \chi(ac\ g_4(x,y))|dxdy|$.

Since that polynomial $g_4(x,y) = x^{18}(1+\mathfrak{p}^{j+1}y)^6[(\mathfrak{p}^{2+2j-2n}y^2(1-c+\mathfrak{p}^{j+1}y)+x^2(1+\mathfrak{p}^{j+1}y)^2]$, we have $\overline{g_4}(x,y) = x^{20}$.

By using equation (A.1.1), so we obtain that,

$$\begin{aligned} I_4 &= \sum_{j=n}^{\infty} q^{-1-j-2ns} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{(a,b)+(\mathfrak{p}O_K)^2} \chi(ac\ g_4(x,y))|dxdy| \\ &= \sum_{j=n}^{\infty} q^{-3-j-2ns} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int_{O_K^2} \chi(ac\ g_4(a+\mathfrak{p}x, b+\mathfrak{p}y))|dxdy|. \end{aligned}$$

Now we apply the change variables (A.1.3) to function g_4 and since that $\frac{\overline{\partial g_4}}{\partial x}(\overline{a},\overline{b}) = 20\overline{a}^{19} \neq 0$, we obtain that,

$$\begin{aligned} I_4 &= \sum_{j=n}^{\infty} q^{-3-j-2ns} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int\limits_{O_K} \chi(ac\ (g_4(a,b)+\mathfrak{p}z_1))|dz_1| \\ &= \sum_{j=n}^{\infty} q^{-3-j-2ns}\overline{I}_4(s,(a,b)), \end{aligned}$$

where $\overline{I}_4(s,(a,b)) = \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2} \int\limits_{O_K} \chi(ac\ (g_4(a,b)+\mathfrak{p}z_1))|dz_1|$, then given that $N = Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2 : \overline{g}_4(\overline{a},\overline{b}) = 0\} = Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2 : \overline{a}^{20} = 0\} = 0$, we obtain

$$\overline{I}_4(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if } \chi=\chi_{triv} \\ \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \overline{\chi}(\overline{a}^{20}) & \textit{if } \overline{\chi}=\chi_{triv} \\ 0 & \textit{all other cases,} \end{cases}$$

where $\overline{\chi}$ the multiplicative character induced by χ in $\mathbb{F}_q$.

Now since that $\overline{\chi}^{20}=\chi_{triv}$ and $\chi|_U=\chi_{triv}$ implies $\chi^{20}=\chi_{triv}$, we get

$$\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \overline{\chi}(\overline{a}^{20}) = \begin{cases} (q-1)^2 & \textit{if } \chi^{20}=\chi_{triv}, \chi|_U=\chi_{triv} \\ 0 & \textit{all other cases,} \end{cases}$$

where $U=1+\mathfrak{p}O_K$.

Then,

$$\overline{I}_4(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if } \chi=\chi_{triv} \\ (q-1)^2 & \textit{if } \chi^{20}=\chi_{triv}, \chi|_U=\chi_{triv} \\ 0 & \textit{all other cases.} \end{cases}$$

Finally since that $I_4=\sum_{j=n}^{\infty} q^{-3-j-2ns}\overline{I}_4(s,(a,b))$, we conclude that

$$I_4 = \begin{cases} q^{-2ns-n-1}(1-q^{-1}) & \chi=\chi_{triv} \\ q^{-2ns-n-1}(1-q^{-1}) & \chi^{20}=\chi_{triv}, \chi|_U=\chi_{triv} \\ 0 & \textit{all other cases.} \end{cases}$$

(e) $I_5=\sum_{j=0}^{2n-2} q^{-1-j-(j+1)s} \int\limits_{O_K^{\times 2}} \chi(ac\ g_5(x,y))|dxdy|$,

where polynomial

$$g_5(x,y)=[x^{18}(c+\mathfrak{p}^{j+1}y)^6][y(c-1+\mathfrak{p}^{j+1}y)^2+\mathfrak{p}^{2n-(1+j)}x^2(c+\mathfrak{p}^{j+1}y)^2)],$$

and $\overline{g_5}(x,y)=x^{18}y^2c^6(c-1)^2$.

By using equation (A.1.1), so we obtain that,

$$I_5 = \sum_{j=0}^{2n-2} q^{-1-j-(j+1)s} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int_{(a,b)+(\mathfrak{p}O_K)^2} \chi(ac\ g_5(x,y))|dxdy|$$
$$= \sum_{j=0}^{2n-2} q^{-3-j-(j+1)s} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int_{O_K^2} \chi(ac\ g_5(a+\mathfrak{p}x, b+\mathfrak{p}y))|dxdy|.$$

Now we apply the change variables (A.1.3) to function g_5 and since that $\frac{\partial \overline{g_5}}{\partial x}(\overline{a},\overline{b}) = 18(\overline{a}^{17})(\overline{b}^2)\overline{c}^6(\overline{c}-1)^2 \neq 0$, we obtain that,

$$I_5 = \sum_{j=0}^{2n-2} q^{-3-j-(1+j)s} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int_{O_K} \chi(ac\ (g_5(a,b)+\mathfrak{p}z_1))|dz_1|$$
$$= \sum_{j=0}^{2n-2} q^{-3-j-(1+j)s}\overline{I}_5(s,(a,b)),$$

where $\overline{I}_5(s,(a,b)) = \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int_{O_K} \chi(ac\ (g_5(a,b)+\mathfrak{p}z_1))|dz_1|$, thus we use Lemma 1.2.2 and give that

$$N = Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2 : \overline{g}_5(\overline{a},\overline{b})=0,\}$$
$$= Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2 : \overline{a}^{18}\overline{b}^2\overline{c}^6(\overline{c}-1)^2=0\},$$
$$=0.$$

$$\overline{I}_5(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if } \ \chi=\chi_{triv} \\ \sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2 \\ \overline{g}_5(\overline{a},\overline{b})\neq 0}} \chi(ac(\overline{g}_5(\overline{a},\overline{b})) & \textit{if } \ \chi|_U=\chi_{triv} \\ 0 & \textit{in other case,} \end{cases}$$

where $U = 1+\mathfrak{p}O_K$.

Now, since that $\overline{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}_q$, we have that

$$\overline{I}_5(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if } \chi=\chi_{triv} \\ \sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2 \\ \overline{a}^{18}\overline{b}^2\overline{c}^6(\overline{c}-1)\neq 0}} \overline{\chi}^{18}(\overline{a})\overline{\chi}^2(\overline{b})\overline{\chi}(\overline{c}^6(\overline{c}-1)^2) & \textit{if } \overline{\chi}=\chi_{triv} \\ 0 & \textit{all other cases.} \end{cases}$$

and given that $\overline{\chi}=\chi_{triv}$ and $\chi|_U=\chi_{triv}$ implies $\chi=\chi_{triv}$, we get

$$\overline{I}_5(s,(a,b)) = \begin{cases} (q-1)^2 & \textit{if } \chi=\chi_{triv} \\ \overline{\chi}(\overline{c}^6(\overline{c}-1)^2)(q-1)^2 & \textit{if } \chi^2=\chi_{triv}, \chi|_U=\chi_{triv} \\ 0, & \textit{all other cases,} \end{cases}$$

Finally, since that $I_5 = \sum_{j=0}^{2n-2} q^{-3-j-(1+j)s}\overline{I}_5(s,(a,b))$, then we conclude that

$$I_5 = \begin{cases} \frac{q^{-1-s}(1-q^{(2n-1)(-1-s)})(1-q^{-1})^2}{1-q^{-1-s}} & \textit{if } \chi=\chi_{triv} \\ \overline{\chi}(\overline{c}^6(\overline{c}-1)^2)\frac{q^{-1-s}(1-q^{(2n-1)(-1-s)})(1-q^{-1})^2}{1-q^{-1-s}}, & \textit{if } \chi^2=\chi_{triv}, \chi|_U=\chi_{triv} \\ 0, & \textit{all other cases.} \end{cases}$$

(f) $I_6 = q^{-2ns-2n} \int_{O_K^{\times 2}} \chi(ac\ g_6(x,y))|g_6(x,y)|^s|dxdy|$.

Since that polynomial $g_6(x,y) = x^{18}(c+\mathfrak{p}^{2n}y)^6[(y(c-1+\mathfrak{p}^{2n})^2+x^2(c+\mathfrak{p}^{2n}y)^2]$, we have $\overline{g_6}(x,y) = x^{18}yc^6(c-1)^2+x^{20}c^8$.

By using similar argument apply in previous cases we get

$$\frac{\partial\overline{g_6}}{\partial x}(\overline{a},\overline{b}) = 2\overline{a}^{17}c^6[9\overline{b}(c-1)^2+10\overline{a}^2c^2]\neq 0$$

and therefore,

$$\begin{aligned}I_6 =& q^{-2ns-2n-2}\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2}\int_{O_K}\chi(ac\ (g_6(a,b)+\mathfrak{p}z_1))|g_6(a,b)+\mathfrak{p}z_1|^s|dz_1|,\\ =& q^{-2ns-2n-2}\overline{I}_6(s,(a,b)),\end{aligned}$$

where $\overline{I}_6=\sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2}\int\limits_{O_K}\chi(ac\ (g_6(a,b)+\mathfrak{p}z_1))|g_6(a,b)+\mathfrak{p}z_1|^s|dz_1|$, then we have

$$\overline{I}_{6,1}=\begin{cases}\frac{q^{-s}(1-q^{-1})N_2}{(1-q^{-1-s})} & \textit{if}\ \ \chi=\chi_{triv}\\ \\ 0 & \textit{all other cases},\end{cases}$$

where

$$\begin{aligned}N_2 =& Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2:\overline{g}_6(\overline{a},\overline{b})=0\},\\ =& Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2:\overline{a}^{18}\overline{b}\overline{c}^6(\overline{c}-1)^2+\overline{a}^{20}\overline{c}^2=0\},\end{aligned}$$

and

$$\overline{I}_{6,2}=\begin{cases}(q-1)^2-N_2 & \textit{if}\ \ \chi=\chi_{triv}\\ \\ \sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2\\ \overline{g}_6(\overline{a},\overline{b})\neq0}}\chi(\overline{g}_6(\overline{a},\overline{b})) & \textit{if}\ \ \chi|_U=\chi_{triv}\\ \\ 0 & \textit{all other cases},\end{cases}$$

where $U=1+\mathfrak{p}O_K$.

Now, since that $\overline{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}_q$, we have that

$$\overline{I}_{6,2}(s,(a,b)) = \begin{cases} (q-1)^2 - N_2 & \textit{if } \chi = \chi_{triv} \\ \sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2 \\ (\overline{b}\overline{c}^4(\overline{c}-1)^2+\overline{a}^2)\neq 0}} \overline{\chi}(\overline{a}^{18}\overline{c}^2(\overline{b}\overline{c}^4(\overline{c}-1)^2+\overline{a}^2)) & \textit{if } \overline{\chi} = \chi_{triv} \\ 0 & \textit{all other cases.} \end{cases}$$

Give that $\overline{\chi} = \chi_{triv}$ and $\chi|_U = \chi_{triv}$ is equivalent to $\chi = \chi_{triv}$, we get

$$\overline{I}_{6,2} = \begin{cases} (q-1)^2 - N_2 + T_3 & \textit{if } \chi = \chi_{triv} \\ 0 & \textit{all other cases,} \end{cases}$$

where $T_3 = \sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2 \\ (\overline{b}\overline{c}^4(\overline{c}-1)^2+\overline{a}^2)\neq 0}} \overline{\chi}(\overline{a}^{18}\overline{c}^2(\overline{b}\overline{c}^4(\overline{c}-1)^2+\overline{a}^2))$,

and since that $I_6 = q^{-2n-2ns-2}(\overline{I}_{6,1}(s,(a,b)) + (\overline{I}_{6,2}(s,(a,b))$, we conclude that

$$I_6 = \begin{cases} q^{-n-2ns-2}\left(\frac{q^{-s}(1-q^{-1})N_2}{(1-q^{-1-s})} + (q-1)^2 - N_2 + T_3\right) & \textit{if } \chi = \chi_{triv} \\ 0 & \textit{all other cases.} \end{cases}$$

(g) $I_7 = \sum_{j=2n}^{\infty} q^{-1-j-2ns} \int_{O_K^{\times 2}} \chi(ac\ g_7(x,y))|dxdy|$.

Since that polynomial $g_7(x,y) = x^{18}(c+\mathfrak{p}^{j+1}y)^6[(\mathfrak{p}^{1+j-2n}y(c-1+\mathfrak{p}^{j+1}y)^2 + x^2(c+\mathfrak{p}^{j+1}y)^2]$, we have $\overline{g_7}(x,y) = x^{20}c^8$.

By using equation (A.1.1), so we obtain that,

$$\begin{aligned} I_7 &= \sum_{j=2n}^{\infty} q^{-1-j-2ns} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int_{(a,b)+(\mathfrak{p}O_K)^2} \chi(ac\ g_7(x,y))|dxdy| \\ &= \sum_{j=2n}^{\infty} q^{-3-j-2ns} \sum_{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2} \int_{O_K^2} \chi(ac\ g_7(a+\mathfrak{p}x, b+\mathfrak{p}y))|dxdy|. \end{aligned}$$

Now we apply the change variables (A.1.3) to function g_7 and since that $\frac{\partial \overline{g_7}}{\partial x}(\overline{a},\overline{b}) = 20c^8\overline{a}^{19} \neq 0$, we obtain that,

$$I_7 = \sum_{y_0 \not\equiv 1, c (mod \mathfrak{p})} \sum_{j=0}^{\infty} q^{-3-j} \sum_{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2} \int_{O_K} \chi(ac\ (g_7(a,b)+\mathfrak{p}z_1))|dz_1|,$$
$$= \sum_{j=2n}^{\infty} q^{-3-j-2ns} \overline{I}_7(s,(a,b)),$$

where $\overline{I_7}(s,(a,b)) = \sum_{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2} \int_{O_K} \chi(ac\ (g_7(a,b)+\mathfrak{p}z_1))|dz_1|$, then by Lemma 1.2.2 and given that

$$N = Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2 : \overline{g}_1(\overline{a},\overline{b}) = 0\} = Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2\} : \overline{a}^{20}\overline{c}^{8} = 0\} = 0,$$

$$\overline{I}_7 = \begin{cases} (q-1)^2 & if\ \ \chi = \chi_{triv} \\ \sum_{\substack{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2 \\ \overline{g}_7(\overline{a},\overline{b}) \neq 0}} \chi(ac(\overline{g}_7(\overline{a},\overline{b})) & if\ \ \chi|_U = \chi_{triv} \\ 0 & all\ other\ cases, \end{cases}$$

with $U = 1+\mathfrak{p}O_K$.

Now, since that $\overline{\chi}$ is the multiplicative character induced by χ in $\mathbb{F}_q$, we have that

$$\overline{I}_7 = \begin{cases} (q-1)^2 & if\ \ \chi = \chi_{triv} \\ \sum_{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2} \overline{\chi}^{20}(\overline{a})\overline{\chi}(\overline{c}^{8}) & if\ \ \overline{\chi} = \chi_{triv} \\ 0 & all\ other\ cases. \end{cases}$$

Now since that $\overline{\chi} = \chi_{triv}$ and $\chi|_U = \chi_{triv}$ is equivalent to $\chi = \chi_{triv}$, we get that

$$\sum_{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2} \overline{\chi}^{20}(\overline{a})\overline{\chi}(\overline{c}^{8}) = \begin{cases} \overline{\chi}(\overline{c}^{8})(q-1)^2 & if\ \chi^{20} = \chi_{triv}, \chi|_U = \chi_{triv} \\ 0 & all\ other\ cases. \end{cases}$$

Furthermore,

$$\overline{I}_7=\begin{cases}(q-1)^2 & \textit{if} \ \ \chi=\chi_{triv}\\ \\ \overline{\chi}(\overline{c}^8)(q-1)^2 & \textit{if}\ \chi^{20}=\chi_{triv},\chi|_U=\chi_{triv}\\ \\ 0 & \textit{all other cases.}\end{cases}$$

Now since that $I_7=\sum_{j=2n}^{\infty}q^{-3-j-2ns}\overline{I}_7(s,(a,b))$, we obtain

$$I_7=\begin{cases}q^{-1-2ns-2n}(1-q^{-1}) & \textit{if} \ \ \chi=\chi_{triv}\\ \\ \overline{\chi}(\overline{c}^8)q^{-1-2ns-2n}(1-q^{-1}) & \textit{if}\ \chi^{20}=\chi_{triv},\chi|_U=\chi_{triv}\\ \\ 0 & \textit{all other cases.}\end{cases}$$

Finally, since that

$Z(s,g,\chi,\Delta_5)=\sum_{n=1}^{\infty}q^{-5n-18ns}I=\sum_{n=1}^{\infty}q^{-5n-18ns}\sum_i I_i$, for $i=1,\cdots,7$, then when $\chi=\chi_{triv}$,

$$\begin{aligned}Z(s,g,\chi,\Delta_5)&=\sum_{n=1}^{\infty}q^{-5n-18ns}\left(\frac{q^{-n-2ns-2-s}(1-q^{-1})N_1}{(1-q^{-1-s})}+(q-1)^2-N_1+T_2\right)\\ &+\sum_{n=1}^{\infty}q^{-5n-18ns}\left(\frac{q^{-2n-2ns-2-s}(1-q^{-1})N_2}{1-q^{-1-s}}+(q-1)^2-N_2+T_3\right)\\ &+\sum_{n=1}^{\infty}q^{-5n-18ns}\left(\frac{q^{-1-2s}(1-q^{(n-1)(-1-2s)})(1-q^{-1})^2}{1-q^{-1-2s}}\right)\\ &+\sum_{n=1}^{\infty}q^{-5n-18ns}\left(\frac{q^{-1-s}-q^{-2n-2ns}(1-q^{-1})^2}{1-q^{-1-s}}\right)\\ &+\sum_{n=1}^{\infty}q^{-5n-18ns}\left(q^{-1}(q-3)(1-q^{-1})+(1-q^{-1})(q^{-2ns-n-1})\right)\\ &+\sum_{n=1}^{\infty}q^{-5n-18ns}(1-q^{-1})(q^{-2ns-2n-1})\end{aligned}$$

Therefore,

$$Z(s,g,\chi,\Delta_5)=\frac{q^{-6-20s}U_0(q^{-s})}{(1-q^{-1-s})(1-q^{-6-20s})}+\frac{q^{-7-20s}U_1(q^{-s})}{(1-q^{-1-s})(1-q^{-7-20s})},$$
$$+\frac{(1-q^{-1})^2q^{-6-20s}}{(1-q^{-1-2s})(1-q^{-5-18s})}-\frac{(1-q^{-1})^2q^{-6-20s}}{(1-q^{-1-2s})(1-q^{-6-20s})}$$
$$+\frac{(1-q^{-1})^2(q^{-6-19s})}{(1-q^{-5-18s})(1-q^{-1-s})}+\frac{(1-q^{-1})^2(q^{-7-20s})}{(1-q^{-7-20s})(1-q^{-1-s})}$$
$$+\frac{(q-3)(1-q^{-1})q^{-6-18s}}{(1-q^{-5-18s})}+\frac{(1-q^{-1})(q^{-7-20s})}{(1-q^{-6-20s})}$$
$$-\frac{(1-q^{-1})(q^{-8-20s})}{(1-q^{-7-20s})}$$

where $U_0(q^{-s})=q^{-2-s}(1-q^{-1})N_1+T_2(1-q^{-1-s})\{(q-1)^2-N_1\}$, with

$N_1=Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2:\overline{a}^{18}(\overline{b}^2(1-\overline{c})+\overline{a}^2)=0\}$ and

$T_2=\sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2\\(\overline{b}^2(1-\overline{c})+a^2)\neq 0}}\chi(\overline{a}^{18}(\overline{b}^2(1-\overline{c})+\overline{a}^2)),$

where, $U_1(q^{-s})=q^{-2-s}(1-q^{-1})N_2+T_3(1-q^{-1-s})\{(q-1)^2-N_2\}$, with

$N_2=Card\{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2:\overline{a}^{18}\overline{b}\overline{c}^6(\overline{c}-1)^2+\overline{a}^{20}\overline{c}^2=0\}$

and

$T_3=\sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q^\times)^2\\(\overline{b}^2(1-\overline{c})+\overline{a}^2)\neq 0}}\overline{\chi}(\overline{a}^{18}(\overline{b}^2(1-\overline{c})+\overline{a}^2)),$

When $\chi\neq\chi_{triv}$ and $\chi|_{1+\mathfrak{p}O_K}=\chi_{triv}$ we have several cases: if $\chi^2=\chi_{triv}$, we have

$$Z(s,g,\chi,\Delta_5)=\sum_{n=1}^{\infty}q^{-5n-18ns}\overline{\chi}(1-\overline{c})\frac{q^{-1-2s}(1-q^{(n-1)(-1-2s)})(1-q^{-1})^2}{1-q^{-1-2s}}$$
$$+\sum_{n=1}^{\infty}q^{-5n-18ns}\overline{\chi}(\overline{c}^6(\overline{c}-1))\frac{q^{-1-s}-q^{-2n-2ns}(1-q^{-1})^2}{1-q^{-1-s}}$$
$$=\overline{\chi}(1-\overline{c})\frac{(1-q^{-1})^2q^{-6-20s}}{(1-q^{-1-2s})(1-q^{-5-18s})}-\overline{\chi}(1-\overline{c})\frac{(1-q^{-1})^2q^{-6-20s}}{(1-q^{-1-2s})(1-q^{-6-20s})}$$
$$+\overline{\chi}(\overline{c}^6(\overline{c}-1)^2)\frac{(1-q^{-1})^2(q^{-6-19s})}{(1-q^{-5-18s})(1-q^{-1-s})}+\overline{\chi}(\overline{c}^6(\overline{c}-1)^2)\frac{(1-q^{-1})^2(q^{-7-20s})}{(1-q^{-7-20s})(1-q^{-1-s})}$$

If $\chi^{18}=\chi_{triv}$, then

$$\begin{aligned}Z(s,g,\chi,\Delta_5)&=\overline{\chi}(\overline{y_0}^7(\overline{y_0}-1))\sum_{n=1}^{\infty}q^{-5n-18ns}q^{-1}(q-3)(1-q^{-1})\\&=\overline{\chi}(\overline{y_0}^7(\overline{y_0}-1))\frac{(q-3)(1-q^{-1})q^{-6-18s}}{(1-q^{-5-18s})}\end{aligned}$$

Finally for $\chi^{20}=\chi_{triv},\quad \chi|_U=\chi_{triv}$, where $U=1+\mathfrak{p}O_K$.

$$\begin{aligned}Z(s,g,\chi,\Delta_5)&=\sum_{n=1}^{\infty}q^{-5n-18ns}(1-q^{-1})(q^{-2ns-n-1})+\\&+\overline{\chi}(\overline{c}^8)\sum_{n=1}^{\infty}q^{-5n-18ns}(1-q^{-1})(q^{-2ns-2n-1})\\&=\frac{(1-q^{-1})(q^{-7-20s})}{(1-q^{-6-20s})}-\overline{\chi}(\overline{c}^8)\frac{(1-q^{-1})(q^{-8-20s})}{(1-q^{-7-20s})}\end{aligned}$$

Summarizing over all cones, we conclude that,

When $\chi=\chi_{triv}$.

$$\begin{aligned}Z(s,f,\chi_{triv})&=2q^{-1}(1-q^{-1})+\frac{q^{-2-6s}(1-q^{-1})}{(1-q^{-2-6s})}\\&+\frac{q^{-7-24s}(1-q^{-1})^2}{(1-q^{-2-6s})(1-q^{-5-18s})}+\frac{q^{-8-27s}(1-q^{-1})^2}{(1-q^{-3-9s})(1-q^{-5-18s})}\\&+\frac{q^{-3-9s}(1-q^{-1})}{(1-q^{-3-9s})}+\frac{q^{-6-20s}U_0(q^{-s})}{(1-q^{-1-s})(1-q^{-6-20s})}\\&+\frac{q^{-7-20s}U_1(q^{-s})}{(1-q^{-1-s})(1-q^{-7-20s})}+\frac{(1-q^{-1})^2q^{-6-20s}}{(1-q^{-1-2s})(1-q^{-5-18s})}\\&-\frac{(1-q^{-1})^2q^{-6-20s}}{(1-q^{-1-2s})(1-q^{-6-20s})}+\frac{(1-q^{-1})^2(q^{-6-19s})}{(1-q^{-5-18s})(1-q^{-1-s})}\\&+\frac{(1-q^{-1})^2(q^{-7-20s})}{(1-q^{-7-20s})(1-q^{-1-s})}+\frac{(q-3)(1-q^{-1})q^{-6-18s}}{(1-q^{-5-18s})}\\&+\frac{(1-q^{-1})(q^{-7-20s})}{(1-q^{-6-20s})}-\frac{(1-q^{-1})(q^{-8-20s})}{(1-q^{-7-20s})}\end{aligned}$$

where

$$U_0(q^{-s}) = q^{-2-s}(1-q^{-1})N_1 + T_2(1-q^{-1-s})\{(q-1)^2 - N_1\},$$
$$N_1 = Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2 : \overline{a}^{18}(\overline{b}^2(1-\overline{c})+\overline{a}^2)=0\},$$
$$T_2 = \sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q)^2 \\ (\overline{b}^2(1-\overline{c})+a^2)\neq 0}} \chi(\overline{a}^{18}(\overline{b}^2(1-\overline{c})+\overline{a}^2)),$$

Furthermore,

$$U_1(q^{-s}) = q^{-2-s}(1-q^{-1})N_2 + T_3(1-q^{-1-s})\{(q-1)^2 - N_2\},$$
$$N_2 = Card\{(\overline{a},\overline{b}) \in (\mathbb{F}_q^{\times})^2 : \overline{a}^{18}\overline{b}\overline{c}^6(\overline{c}-1)^2+\overline{a}^{20}\overline{c}^2=0\},$$
$$T_3 = \sum_{\substack{(\overline{a},\overline{b})\in(\mathbb{F}_q^{\times})^2 \\ (\overline{b}^2(1-\overline{c})+\overline{a}^2)\neq 0}} \overline{\chi}(\overline{a}^{18}(\overline{b}^2(1-\overline{c})+\overline{a}^2)),$$

$\chi \neq \chi_{triv}$ and $\chi|_{1+\mathfrak{p}O_K} = \chi_{triv}$ we have several cases: if $\chi^2 = \chi_{triv}$, we have

$$Z(s,g,\chi) = \overline{\chi}(1-\overline{c})\frac{(1-q^{-1})^2q^{-6-20s}}{(1-q^{-1-2s})(1-q^{-5-18s})}$$
$$-\overline{\chi}(1-\overline{c})\frac{(1-q^{-1})^2q^{-6-20s}}{(1-q^{-1-2s})(1-q^{-6-20s})} + \overline{\chi}(\overline{c}^6(\overline{c}-1)^2)\frac{(1-q^{-1})^2(q^{-6-19s})}{(1-q^{-5-18s})(1-q^{-1-s})}$$
$$+\overline{\chi}(\overline{c}^6(\overline{c}-1)^2)\frac{(1-q^{-1})^2(q^{-7-20s})}{(1-q^{-7-20s})(1-q^{-1-s})}$$

In the case where $\chi^6=\chi_{triv}$.

$$Z(s,f,\chi)=\chi(-\overline{c})\left(q^{-1}(1-q^{-1})+\frac{q^{-3-6s}(1-q^{-1})}{(1-q^{-2-6s})}+\frac{q^{-2-6s}(1-q^{-1})^2}{(1-q^{-2-6s})}\right)$$
$$+\chi(-\overline{c})\left(\frac{q^{-7-24s}(1-q^{-1})^2}{(1-q^{-2-6s})(1-q^{-5-18s})}\right).$$

If $\chi^9=\chi_{triv}$.

$$Z(s,f,\chi,\Delta_i)=\frac{q^{-8-27s}(1-q^{-1})^2}{(1-q^{-3-9s})(1-q^{-5-18s})}$$
$$+\frac{q^{-3-9s}(1-q^{-1})^2}{(1-q^{-3-9s})}+\frac{q^{-4-9s}(1-q^{-1})}{(1-q^{-3-9s})}+q^{-1}(1-q^{-1}).$$

In the case where $\chi^{18}=\chi_{triv}$.

$$Z(s,g,\chi,\Delta_5)=\overline{\chi}(\overline{y_0}^7(\overline{y_0}-1))\frac{(q-3)(1-q^{-1})q^{-6-18s}}{(1-q^{-5-18s})}$$

Finally for $\chi^{20}=\chi_{triv}$.

$$Z(s,g,\chi,\Delta_5)=\frac{(1-q^{-1})(q^{-7-20s})}{(1-q^{-6-20s})}-\overline{\chi}(\overline{c}^8)\frac{(1-q^{-1})(q^{-8-20s})}{(1-q^{-7-20s})}$$

In all other cases, $Z(s,f,\chi,\Delta_i)=0$.

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