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On the separability of the partial skew groupoid ring

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Abstract

Given a partial action α of a groupoid $\mathcal{G}$ on a unital K-algebra $\mathcal{A}$, we establish necessary and sufficient conditions for which the extension $\mathcal{A} \star_\alpha \mathcal{G} \supset \mathcal{A}$ is separable.

Key words & phrases: Partial action of groupoid, skew groupoid ring, separable extension.

1 Introduction

The notion of partial groupoid action that we use in this work was introduced by D. Bagio and A. Paques in [2], which is a natural extension of the concept of partial group action given by M. Docuchaev and R. Exel in [3].

Let $\mathcal{G}$ be a groupoid, K a commutative ring and $\mathcal{A}$ a unital K-algebra. A partial action of $\mathcal{G}$ on $\mathcal{A}$ is a pair

$$\alpha = (\{\mathcal{A}_g\}_{g\in\mathcal{G}}, \{\alpha_g\}_{g\in\mathcal{G}}),$$

where for each $g \in \mathcal{G}$, $\mathcal{A}_g$ is an ideal of $\mathcal{A}_{r(g)}$, $\mathcal{A}_{r(g)}$ is an ideal of $\mathcal{A}$, $\alpha_g : \mathcal{A}_{g^{-1}} \to \mathcal{A}_g$ is an K-isomorphism, and the following conditions hold:

(i) $\alpha_e = id_{\mathcal{A}_e}$ is the identity map of $\mathcal{A}_e$ for any identity e of $\mathcal{G}$,

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(ii) $\alpha_h^{-1}(\mathcal{A}_{g^{-1}} \cap \mathcal{A}_h) \subseteq \mathcal{A}_{(gh)^{-1}}$,

(iii) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$, for each $x \in \alpha_h^{-1}(\mathcal{A}_g^{-1} \cap \mathcal{A}_h)$ and $g, h \in \mathcal{G}$ such that gh exists.

Given such an action α of $\mathcal{G}$ on $\mathcal{A}$, according to [1] the partial skew groupoid ring $\mathcal{A} \star_\alpha \mathcal{G}$ is defined by the direct sum

$$\mathcal{A} \star_\alpha \mathcal{G} = \bigoplus_{g \in \mathcal{G}} \mathcal{A}_g \delta_g,$$

where the δ_g's are symbols, with the usual addition, and multiplication determined by the rule

$$(a_g\delta_g)(b_h\delta_h) = \begin{cases} \alpha_g(\alpha_{g^{-1}}(a_g)b_h)\delta_{gh}, & \text{if } (g,h) \in \mathcal{G}^2 \\ 0, & \text{otherwise} \end{cases},$$

for all $g, h \in \mathcal{G}$, $a_g \in \mathcal{A}_g$ y $b_h \in \mathcal{A}_h$.

It is possible to show that this multiplication is well defined, and that $\mathcal{A} \star_\alpha \mathcal{G}$ is an associative algebra, but in general it has no unity, [1].

A ring extension $S \supset R$ is called separable, if S is a direct sum of $S \bigotimes_R S$ as an (R,R)-bimodule, which is equivalent to say that there exists an element $x \in S \bigotimes_R S$ which is S-central (i.e., $xs = sx$ for all $s \in S$) and satisfies the condition $m_S(x) = 1_s$, see [4].

If $\mathcal{G}$ is a finite groupoid, we characterize when the extension $\mathcal{A} \supset \mathcal{A} \star_\alpha \mathcal{G}$ is separable. Furthermore, we show that such a condition on $\mathcal{G}$ no possible removed it.

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